Problem 11. The following problem outlines a proof of Theorem 1.6.7 that is different from a straightforward generalization of the proof of Theorem 1.6.2. Let V be a vector space with basis $\{e_1, \dots, e_n\}$.

(a) For a fixed tuple $I = (i_1, ..., i_r)$ with $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, use multilinear extension to conclude that there exists a multilinear map $\varphi_I \colon V^r \to K$ satisfying

$$\varphi_I(e_{j_1}, \dots, e_{j_r}) = \begin{cases} \operatorname{sgn}(\sigma) & \text{if there exists a permutation } \sigma \text{ of [r] s.t. } j_{\sigma(k)} = i_k \\ & \text{for all } k \in [r]; \\ 0 & \text{otherwise.} \end{cases}$$

for all $(j_1, \dots, j_r) \in [n]^r$.

(b) Show that φ_I is alternating, and conclude that it therefore factors through $\bigwedge^r \colon V^r \to \bigwedge^r V$ to yield a linear map $\overline{\varphi_I} \colon \bigwedge^r V \to K$ satisfying $\overline{\varphi_I} \circ \bigwedge^r = \varphi_I$. (Careful: it is not sufficient to check the alternating property on tuples of basis vectors!)

(c) Show that the set $\{ e_{j_1} \wedge \cdots \wedge e_{j_r} : 1 \leq j_1 < j_2 < \cdots < j_r \leq n \}$ is linearly independent. (Consider a linear relation of these elements and apply suitable maps $\overline{\varphi_I}$ to conclude all the coefficients are zero.)

Solution (a) Apply the multilinear extension theorem (Theorem 1.1.4); you only have to observe that the permutation σ is indeed unique, and so $\operatorname{sgn}(\sigma)$ is well-defined.

(b) The crux is to show that φ_I is alternating. The proof is essentially the same as when one shows that the Leibniz formula is alternating (typically checked when one shows existence of the determinant in Linear Algebra).

Let $(v_1, \dots, v_r) \in V^r$ and suppose $k \neq l$ are such that $v_k = v_l$. Express each $v_i = \sum_{j=1}^n \alpha_{i,j} e_i$ with $\alpha_{i,j} \in K$ in terms of the given basis. Then

$$\begin{aligned} \varphi_I\left(\sum_{j_1=1}^n \alpha_{1,j_1} e_{j_1}, \dots, \sum_{j_r=1}^n \alpha_{r,j_k} e_{j_k}\right) &= \sum_{j_1=1}^n \cdots \sum_{j_r=1}^n \alpha_{1,j_1} \dots \alpha_{r,j_k} \varphi_I(e_{j_1}, \dots, e_{j_k}) \\ &= \sum_{(j_1,\dots,j_r) \in [n]^r} \alpha_{1,j_1} \dots \alpha_{r,j_k} \varphi_I(e_{j_1}, \dots, e_{j_k}) = (*) \end{aligned}$$

by multilinearity of φ_I. For $(j_1, \dots, j_r)$ observe: if the entries in the tuple are *not* a permutation of the r distinct numbers $i_1, \dots, i_r$, then $\varphi_I(e_{j_1}, \dots, e_{j_k}) = 0$ (i.e., if either there are duplicate entries in the tuple or if there are some entries that do not appear in $\{i_1, \dots, i_r\}$). So the only summands we are left to consider, are those where $(j_1, \dots, j_r)$ is a permutation of $(i_1, \dots, i_r)$, i.e., where there exists $\sigma \in \mathfrak{S}_r$ such that $j_\nu = i_{\sigma(\nu)}$ for all

$\nu \in [r]$. Then

$$(*) = \sum_{\sigma\in\mathfrak{S}_r} \alpha_{1,i_{\sigma(1)}} \dots \alpha_{r,i_{\sigma(r)}} \varphi_I(e_{i_{\sigma(1)}}, \dots, e_{i_{\sigma(r)}}) = \sum_{\sigma\in\mathfrak{S}_r} \alpha_{1,i_{\sigma(1)}} \dots \alpha_{r,i_{\sigma(r)}} \operatorname{sgn}(\sigma^{-1}).$$
$$= \sum_{\sigma\in\mathfrak{S}_r} \alpha_{1,i_{\sigma(1)}} \dots \alpha_{r,i_{\sigma(r)}} \operatorname{sgn}(\sigma).$$

Now recall our assumption $v_k = v_l$, i.e., $\alpha_{k,j} = \alpha_{l,j}$ for all $j \in [n]$. Let $\tau = (k\ l)$ (transposition of k and l). Then $\alpha_{1,i_{\sigma(1)}} \cdots \alpha_{r,i_{\sigma(r)}} = \alpha_{1,i_{\sigma\tau(1)}} \cdots \alpha_{r,i_{\sigma\tau(r)}}$ (because $\sigma\tau(k) = \sigma(l)$, $\sigma\tau(l) = \sigma(k)$ and $\sigma\tau(\nu) = \sigma\nu$ for $\nu \notin \{k, l\}$).

The group $\mathfrak{S}_r$ is the disjoint union of the two cosets A_r and $A_r\tau$ (here $A_r \subseteq \mathfrak{S}_r$ is the alternating group on r elements, i.e., the subgroup of all permutations σ with $\operatorname{sgn}(\sigma) = 1$).

We split

$$\begin{aligned}(*) &= \sum_{\sigma\in A_r} \alpha_{1,i_{\sigma(1)}} \dots \alpha_{r,i_{\sigma(r)}} \varphi_I(e_{i_{\sigma(1)}}, \dots, e_{i_{\sigma(r)}}) + \sum_{\sigma\in A_r} \alpha_{1,i_{\sigma\tau(1)}} \dots \alpha_{r,i_{\sigma\tau(r)}} \varphi_I(e_{i_{\sigma\tau(1)}}, \dots, e_{i_{\sigma\tau(r)}})\\
&= \sum_{\sigma\in A_r} \alpha_{1,i_{\sigma(1)}} \dots \alpha_{r,i_{\sigma(r)}} (\varphi_I(e_{i_{\sigma(1)}}, \dots, e_{i_{\sigma(r)}}) + \varphi_I(e_{i_{\sigma\tau(1)}}, \dots, e_{i_{\sigma\tau(r)}}))\\
&= \sum_{\sigma\in A_r} \alpha_{1,i_{\sigma(1)}} \dots \alpha_{r,i_{\sigma(r)}} (\operatorname{sgn}\sigma + \operatorname{sgn}\sigma\tau).\\
&= \sum_{\sigma\in A_r} \alpha_{1,i_{\sigma(1)}} \dots \alpha_{r,i_{\sigma(r)}} (\operatorname{sgn}\sigma - \operatorname{sgn}\sigma) = 0.\end{aligned}$$

Now that we know that φ_I is alternating, the universal property of $\bigwedge^r$ implies the existence of $\overline{\varphi_I}$ (Theorem 1.6.5).

(c) Let $\mathcal{M}$ denote the set of all r-element subsets of $[n]$. For $J \in \mathcal{M}$, with $J = \{1 \le j_1 < j_2 < \cdots < j_r \le n\}$, let $e_J := e_{j_1} \wedge \cdots \wedge e_{j_r}$. Let $\lambda_J \in K$ such that

$$\sum_{J\in\mathcal{M}} \lambda_J e_J = 0.$$

For $I \in \mathcal{M}$, consider the map $\overline{\varphi_I}$ from (b) and apply it to the linear combination. Then

$$0 = \overline{\varphi_I}\left(\sum_{J\in\mathcal{M}} \lambda_J e_J\right) = \sum_{J\in\mathcal{M}} \lambda_J \overline{\varphi}_I(e_J) = \lambda_I$$

(because $\lambda_I(e_J) = 0$ if $I \neq J$). Since I was arbitrary, $\lambda_I = 0$ for all $I \in \mathcal{M}$, and therefore the family $(e_J)_{J\in\mathcal{M}}$ is linearly independent.

Problem 13. For a ring A, the center is defined as

$$Z(A) := \{ x \in A : \forall y \in A : xy = yx \}.$$

Show:

(a) $Z(A)$ is a ring.

(b) If R is a commutative ring, then the ring A is a (unitary) associative R-algebra if and only if there exists a ring homomorphism $\varphi\colon R \to Z(A)$.

Solution. (a) Since A is a ring, it suffices to check that $Z(A)$ is a subring, i.e., it suffices to show: $1 = 1_A \in Z(A)$ and for all $x, y \in Z(A)$ also $x + y \in Z(A)$ and $xy \in Z(A)$.

Let $a \in A$ and $x, y \in Z(A)$. Then $1a = a = a1$, so $1 \in Z(A)$. Next, $(x + y)a = xa + ya = ax + ay = a(x + y)$, so $x + y \in Z(A)$ (we used that x, y are central in the middle equality). Finally $(xy)a = x(ya) = x(ay) = (xa)y = (ax)y = a(xy)$, so $xy \in Z(A)$ (the equalities where it is used that x or y are central, are highlighted in red).

(b) For didactic purposes, I write $\bullet$ for the multiplication on A, to make it easier to distinguish it from the R-module structure on A..

Suppose first that A is an unitary associative R-algebra, i.e., the ring A also has an R-module structure satisfying $r(a \bullet b) = (ra) \bullet b = a \bullet (rb)$ for all $r \in R$, $a, b \in A$. Define $\varphi\colon R \to A$ by $r \mapsto r1_A$. We check that $\varphi(R) \subset Z(A)$: Let $a \in A$ and $r \in R$. Then $\varphi(r) \bullet a = (r1_A) \bullet a = r(1_A \bullet a) = r(a \bullet 1_A) = a \bullet (r1_A) = a \bullet \varphi(r)$. Therefore we can restrict the codomain of φ to obtain $\varphi\colon R \to A$ with $\varphi(r) = r1_A$.

Now we have to check that φ is a ring homomorphism. Indeed, $\varphi(1_R) = 1_R 1_A = 1_A$ (this is an axiom of the module structure). Let $r, s \in R$. Then $\varphi(r + s) = (r + s)1_A = r1_A + s1_A$ (distributivity of the module structure), and $\varphi(rs) = (rs)1_A = r(s1_A) = r(s(1_A \bullet 1_A)) = r(1_A \bullet s1_A) = (r1_A) \bullet (s1_A)$ (the second equality is the associativity of the module structure; $1_A \bullet 1_A = 1_A$ because 1_A is the multiplicative identity in A). Thus, the first direction is shown.

Conversely, suppose A is a ring and $\varphi\colon R \to Z(A)$ is a ring homomorphism. Define $R \times A \to A$ by $(r, a) \mapsto ra := \varphi(r) \bullet a$.

We first check that this turns A into an R-module. We already know that $(A, +)$ is an abelian group. Let $r, s \in R$ and $a, b \in A$. Then $1_R a = \varphi(1_R) \bullet a = 1_A \bullet a = a$. Next $(rs)a = \varphi(rs) \bullet a = (\varphi(r) \bullet \varphi(s)) \bullet a = \varphi(r) \bullet (\varphi(s) \bullet a) = \varphi(r) \bullet (sa) = r(sa)$. Finally $(r + s)a = \varphi(r + s) \bullet a = (\varphi(r) + \varphi(s)) \bullet a = \varphi(r) \bullet a + \varphi(s) \bullet a = ra + sa$ and $r(a + b) = \varphi(r) \bullet (a + b) = \varphi(r) \bullet a + \varphi(r) \bullet b = ra + rb$.

To see that A is an R-algebra, we still have to check: for all $a, b \in A$ and $r \in R$, it holds that $r(a \bullet b) = (ra) \bullet b = a \bullet (rb)$. We check the first equality: $r(a \bullet b) = \varphi(r) \bullet (a \bullet b) = (\varphi(r) \bullet a) \bullet b = (ra) \bullet b$. For the second one, we actually need that φ maps into the center: $(ra) \bullet b = (\varphi(r) \bullet a) \bullet b = a \bullet (\varphi(r) \bullet b) = a \bullet (rb)$.