Problem 34. Let A be a finite set with subsets $A_1, \dots, A_n$, and let $d_1, \dots, d_n \geq 1$. Show: If

$$\left|\bigcup_{i\in I} A_i\right| \geq \sum_{i\in I} d_i$$

for all $I \subseteq [n]$, then there exist pairwise disjoint subsets $D_k \subseteq A_k$ with $|D_k| = d_k$.

Hint: Construct a bipartite graph in which A is one side, and the other side consists of vertices corresponding to the sets A_i with suitable multiplicity. Define the edge set of the graph so that the desired result can be derived from Hall's theorem.

Problem 35 (Sperner's Lemma). Let M be a finite set of cardinality n. The power set $\mathcal{P}(M)$ is partially ordered by set inclusion. A subset $\mathcal{X} \subseteq \mathcal{P}(M)$ is

- a *chain* if it is totally ordered (for all $A, B \in \mathcal{X}$ we have $A \subseteq B$ or $B \subseteq A$);
- an *antichain* if no two distinct elements of $\mathcal{X}$ are comparable, i.e.,

$$\forall A, B \in \mathcal{X}: \quad A \neq B \implies A \not\subseteq B \text{ and } B \not\subseteq A.$$

It is clear that the maximal size of a chain in $\mathcal{P}(M)$ is n. Prove that the maximal size of an antichain in $\mathcal{P}(M)$ is $\binom{n}{\lfloor n/2 \rfloor}$.

It may be useful to proceed using the following steps.

(a) Find an antichain of size $\binom{n}{\lfloor n/2 \rfloor}$.

(b) If $\mathcal{P}(M)$ is a union of r chains, then an antichain in $\mathcal{P}(M)$ has size at most r.

(c) Let $G = (\mathcal{P}(M), E)$ be the graph where there is an edge between A and B if and only if either $A \subseteq B$ and $|B \setminus A| = 1$ or, symmetrically, $B \subseteq A$ and $|A \setminus B| = 1$.

Let $\mathcal{X}_k$ denote the subsets of M of cardinality k (for $0 \leq k \leq n$) and let $G_k := G[\mathcal{X}_k \cup \mathcal{X}_{k+1}]$ denote the induced subgraph (for $0 \leq k \leq n-1$). Observe that G_k is bipartite.

Show that G_k has a matching that saturates $\mathcal{X}_k$ if $k < n/2$, and that G_k has a matching that saturates $\mathcal{X}_{k+1}$ if $k + 1 > n/2$. [1]

(d) Connect the matchings to write $\mathcal{P}(M)$ as a union of (pairwise disjoint) chains; argue that there are at most $\binom{n}{\lfloor n/2 \rfloor}$ such chains.

[1] In fact, by considering complements in M, it suffices to prove one of these statements.