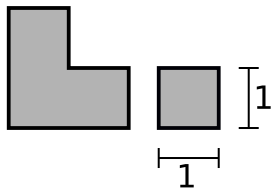
Problem 26. Use the identity $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \in \mathbb{C}[\![z]\!]$, and basic operations for power series (sum, product, differentation, integration) to find closed form expressions for the following power series.

$$\text{(i)} \sum_{n\geq 1} n^2 z^n \qquad \text{(ii)} \sum_{n\geq 0} \frac{n}{n+1} z^n \qquad \text{(iii)} \sum_{n\geq 0} \Big(\sum_{k=1}^{n} \frac{1}{k} \Big) z^n.$$

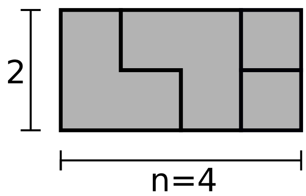
Problem 27. Solve the following recurrence (i.e., find a closed form for a_n):

$$a_0 = 0, a_1 = 0, a_2 = 1 \quad \text{and} \quad a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3} \quad \text{for } n \geq 3.$$

Problem 28. How many ways are there to fill completely without overlap a $2 \times n$ rectangle with pieces of the following types? The sides of the pieces are 1 and 2; the pieces can be rotated by a multiple of a right angle.



An example for $n = 4$ is shown below.



Let a_n denote the number of such tilings of the $2 \times n$ rectangle.

(a) Find a recursion for the sequence $(a_n)_{n\geq 1}$ and find a closed form expression for its (ordinary) generating function.

(b) Determine the asymptotic growth of $(a_n)_{n\geq 1}$.