Problem 22. Construct $f, g\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that $f(n) \neq O(g(n))$ and $g(n) \neq O(f(n))$. Can you make f and g non-decreasing?

Problem 23. Let $f, g, h\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ be such that

$$f(n) = O(h(n)), \qquad g(n) = O(h(n)), \qquad \text{and } h(n) = o(1).$$

Show that $f(n) + g(n) = O(h(n))$, that $f(n)g(n) = o(h(n))$, and that

$$\frac{1}{1 + f(n)} = 1 + O(h(n)).$$

Problem 24. (a) Verify that

$$\binom{1/2}{n} = \frac{(-1)^{n-1}}{4^n(2n-1)}\binom{2n}{n} \qquad \text{for } n \geq 0.$$

(b) Show

$$\binom{2n}{n} = \frac{4^n}{\sqrt{\pi n}}\Big(1 + O\big(\tfrac{1}{n}\big)\Big).$$

(Problem 23 is useful.)

Problem 25. Let $A(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[\![z]\!]$ be a formal power series.

(a) Show that $A(z)$ has a reciprocal if and only if $a_0 \neq 0$.

(b) Show that if $a_0 = 0$, then $1 - A(z)$ has a reciprocal given by

$$B(z) = \sum_{n=0}^{\infty} A(z)^k = 1 + A(z) + A(z)^2 + A(z)^3 + \ldots$$

Also argue why the stated expression for $B(z)$ makes sense in $\mathbb{C}[\![z]\!]$.