Recall that a module M over a ring R is *simple* if it is nonzero and its only submodules are 0 and M itself.

Problem 20. Let R be a ring. An R-module M is *cyclic* if there exists some $m \in M$ such that $M = \langle m \rangle$. Show that

(a) An R-module M is cyclic if and only if $M \cong R/I$ for some left ideal I of R.

(b) Simple R-modules are cyclic; furthermore an R-module M is simple if and only if $M \cong R/I$ for some maximal left ideal I of R.

(c) Every quotient of a cyclic module is cyclic.

Problem 21. Let R be a ring and M an R-module. We first make some definitions.

- A *submodule series* for M is any finite chain of submodules of the form
$$0 = A_0 \leq A_1 \leq \cdots \leq A_n = M. \tag{1}$$
- A *composition series* for M is a submodule series such that each factor A_i/A_{i-1} is a simple module. In that case n is called the *length* of the composition series.
- The module M has *finite length* if there exists a composition series for M.
- A *refinement* of the series in (1) is a submodule series that includes all the A_i, i.e., a submodule series
$$0 = B_0 \leq B_1 \leq \cdots \leq B_m = M.$$
such that each A_i appears in the list $B_0, \ldots, B_m$.
- Two submodule series
$$0 = A_0 \leq A_1 \leq \cdots \leq A_n = M \qquad \text{and} \qquad 0 = B_0 \leq B_1 \leq \cdots \leq B_m = M$$
are *equivalent* if $n = m$ and there exists a permutation σ of $[n]$ such that $A_i/A_{i-1} \cong B_{\sigma(i)}/B_{\sigma(i)-1}$ for all $i \in [n]$.

Prove the following.

(a) (Schreier refinement theorem) Any two submodule series for a module M have equivalent refinements. (Hint: consider modules $A_{ij} = (A_i + B_j) \cap A_{i+1}$ and $B_{ji} = (B_j + A_i) \cap B_{j+1}$ and use the Zassenhaus lemma, Theorem 2.4.13.)

(b) (Jordan-Hölder theorem) If M has finite length, then any two composition series are equivalent. (Hint: use (a).)

Furthermore, find composition series for $\mathbb{Z}/12\mathbb{Z}$, $\mathbb{Z}/9\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. (Can you see what the composition factors of $\mathbb{Z}/n\mathbb{Z}$ would be?)