Problem 17. Let K be a field and let

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0$$

be an exact sequence of (finite-dimensional) vector spaces over K. Show that

$$\sum_{i=1}^{n} (-1)^i \dim(V_i) = 0.$$

Problem 18. Let R be a ring. Consider the following commutative diagram of R-modules and R-morphisms.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d, "\alpha'"] & B' \arrow[r, "g'"] \arrow[d, "\beta'"] & C' \arrow[r] \arrow[d, "\gamma'"] & 0 \\
0 \arrow[r] & A'' \arrow[r, "f''"] \arrow[d] & B'' \arrow[r, "g''"] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Assume that the three rows are exact and that the two columns on the right are exact. Prove that then the left column is also exact.

Problem 19. Let R be a ring. If

$$0 \longrightarrow B \xrightarrow{f} E \xrightarrow{g} A \longrightarrow 0$$

is a short exact sequence of R-modules, the triple (f, E, g) is called an *extension of A by B*.

(a) Prove that, for any two R-modules A, B, at least one extension of A by B exists.

(b) Two extensions (f_1, E_1, g_1) and (f_2, E_2, g_2) of A and B are *equivalent* if there exists an R-morphism $h\colon E_1 \to E_2$ such that $h \circ f_1 = f_2$ and $g_2 \circ h = g_1$. Prove that such an R-morphism h is an isomorphism.

(c) Show that the following two are non-equivalent short exact sequences

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_4 \longrightarrow 0$$
$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_8 \longrightarrow \mathbb{Z}_4 \longrightarrow 0,$$

i.e., that these are extensions of $\mathbb{Z}_4$ by $\mathbb{Z}_2$ that are not equivalent. (Here $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$.)