Problem 9. Identify $\bigwedge^2(\mathbb{R}^3)$ with $\mathbb{R}^3$ by identifying $e_1 \wedge e_2$ with e_3, $e_2 \wedge e_3$ with e_1 and $e_3 \wedge e_1$ with e_2. Show that under this identification, the exterior product $v \wedge w \in \bigwedge^2(\mathbb{R}^3) = \mathbb{R}^3$ is the same as the cross product $v \times w \in \mathbb{R}^3$.

Problem 10. Let K be a field and let $A = (a_{i,j})_{i,j\in[n]}$ be an $n \times n$-matrix with entries in K. Denote by $e_1, \dots, e_n$ the standard basis of K^n. Show that

$$(Ae_1) \wedge \cdots \wedge (Ae_n) = \sum_{\pi \in \mathcal{S}_n} \operatorname{sgn}(\pi) a_{1,\pi(1)} \cdots a_{n,\pi(n)}\, e_1 \wedge \cdots \wedge e_n = \det(A) e_1 \wedge \cdots \wedge e_n.$$

Problem 11. The following problem outlines a proof of Theorem 1.6.7 that is different from a straightforward generalization of the proof of Theorem 1.6.2. Let V be a vector space with basis $\{e_1, \dots, e_n\}$.

(a) For a fixed tuple $I = (i_1, ..., i_r)$ with $1 \le i_1 < i_2 < \cdots < i_r \le n$, use multilinear extension to conclude that there exists a multilinear map $\varphi_I \colon V^r \to K$ satisfying

$$\varphi_I(e_{j_1}, \dots, e_{j_r}) = \begin{cases} \operatorname{sgn}(\sigma) & \text{if there exists a permutation } \sigma \text{ of [r] s.t. } j_{\sigma(k)} = i_k \\ & \text{for all } k \in [r]; \\ 0 & \text{otherwise.} \end{cases}$$

for all $(j_1, \dots, j_r) \in [n]^r$.

(b) Show that φ_I is alternating, and conclude that it therefore factors through $\bigwedge^r \colon V^r \to \bigwedge^r V$ to yield a linear map $\overline{\varphi_I} \colon \bigwedge^r V \to K$ satisfying $\overline{\varphi_I} \circ \bigwedge^r = \varphi_I$. (Careful: it is not sufficient to check the alternating property on tuples of basis vectors!)

(c) Show that the set $\{ e_{j_1} \wedge \cdots \wedge e_{j_r} : 1 \le j_1 < j_2 < \cdots < j_r \le n \}$ is linearly independent. (Consider a linear relation of these elements and apply suitable maps $\overline{\varphi_I}$ to conclude all the coefficients are zero.)