We discuss Problem 6 *from Week 2; in addition the following problems. Again, all vector spaces are finite-dimensional.*

Problem 7. Let U, V be vector spaces.

(a) Let $u_1, \ldots, u_n \in U$ be linearly independent. For $x \in U \otimes V$ having a representation $x = \sum_{i=1}^{n} u_i \otimes v_i$ with $v_1, \ldots, v_n \in V$, show that $v_1, \ldots, v_n \in V$ are uniquely determined.

(b) Show that there exists an isomorphism of vector spaces $\varphi\colon U \otimes V^* \to \operatorname{Hom}(V, U)$ satisfying $\varphi(u \otimes f)(v) = f(v)u$ for all $u \in U$, $f \in V^*$, and $v \in V$. [1]

Problem 8. Let $(V_i)_{i \in I}$ be a family of vector spaces with a finite index set I. Recall that the (*external*) *direct sum* is

$$\bigoplus_{i \in I} V_i = \{\, (v_i)_{i \in I} : v_i \in V_i \,\}$$

with the componentwise vector space structure. For every $k \in I$, there is a monomorphism $\varepsilon_k\colon V_k \to \bigoplus_{i \in I} V_i,\ v \mapsto (0, \ldots, 0, v, 0, \ldots, 0)$ whose only nonzero entry is in the k-th component.

(a) Show that $\bigoplus_{i \in I} V_i$ (together with $(\varepsilon_i)_{i \in I}$) satisfies the following universal property: If W is a vector space and $(f_i \in \operatorname{Hom}(V_i, W))_{i \in I}$ are homomorphisms, then there exists a unique homomorphism $g\colon \bigoplus_{i \in I} V_i \to W$ such that $g \circ \varepsilon_i = f_i$ for all $i \in I$.

(b) Let $(U_j)_{j \in J}$ be another family of vector spaces. Show that there is a canonical isomorphism[2]

$$\Big(\bigoplus_{j \in J} U_j\Big) \otimes \Big(\bigoplus_{i \in I} V_i\Big) \cong \bigoplus_{i \in I, j \in J} (U_j \otimes V_i).$$

[1] The map φ is injective even for infinite-dimensional vector spaces, but its image consists of those linear maps $g \in \operatorname{Hom}(V, U)$ having finite-dimensional image (i.e., $\dim g(V) < \infty$).

[2] The notion *canonical* is used here informally, and should mean that the isomorphism is given in some sense by a simple/natural/canonical formula. The notion can be made precise with concepts from category theory, cf. *natural isomorphism of functors*.

This isomorphism actually also works for infinite-dimensional vector spaces and arbitrary index sets I, J. The external direct-sum then only consists of those tuples where all but finitely many components are zero.