All vector spaces are finite-dimensional and considered over some fixed field K.

Problem 4. Let V be a vector space. Let $A \in K_{m\times m}$ (a $m \times m$ matrix) and suppose $AA^T = I_m$ (here A^T denotes the transpose and I_m the $m \times m$ identity matrix). Let v_1, $\dots, v_m \in K$ and for $j \in [m]$ let $u_j \coloneqq \sum_{i=1}^m a_{ij} v_i$. Prove that

$$\sum_{i=1}^m u_i \otimes u_i = \sum_{i=1}^m v_i \otimes v_i.$$

Problem 5. Let U, V be vector spaces.

(a) Let $x \in U \otimes V$. Suppose that $k \geq 0$ is minimal such that there exist $u_1, \dots, u_k \in U$ and $v_1, \dots, v_k \in V$ with $x = \sum_{i=1}^k u_i \otimes v_i$. Show that $(u_1, \dots, u_k)$, respectively $(v_1, \dots, v_k)$, are linearly independent.

(b) Suppose that $e_1, e_2 \in V$ are linearly independent. Prove that $e_1 \otimes e_2 + e_2 \otimes e_1 \in V \otimes V$ is indecomposable.

Problem 6. Let U, V, W be vector spaces. Recall that $M(U, V, W)$ denotes the vector space of bilinear maps $U \times V \to W$.

(a) Show that there is an isomorphism of vector spaces (i.e., a bijective linear map)

$$F\colon M(U, V, W) \to \operatorname{Hom}(U \otimes V, W),$$

satisfying $F(\varphi)(u \otimes v) = \varphi(u, v)$ for all $\varphi \in M(U, V, W)$, $u \in U$, $v \in V$.

(b) Show that there is an isomorphism of vector spaces

$$G\colon \operatorname{Hom}(U, \operatorname{Hom}(V, W)) \to M(U, V, W),$$

satisfying $G(T)(u, v) = T(u)(v)$ for all $T \in \operatorname{Hom}(U, \operatorname{Hom}(V, W))$, $u \in U$, $v \in V$.