*The first problem sheet is **optional**. Throughout, vector spaces are considered over an arbitrary but fixed field K. $V_1, \dots, V_m, W_1, \dots, W_m, P, U, V, W$, are vector spaces. (Keep in mind that all vector spaces considered are implicitly assumed to be finite-dimensional.)*

Problem 1. Let $\varphi\colon V_1 \times \cdots \times V_m \longrightarrow W$ be multilinear and let $T_i\colon W_i \longrightarrow V_i$ (for $i \in [m]$) and $T\colon W \longrightarrow V$ be linear.

(a) The map $\psi := \varphi \circ (T_1, \dots, T_m)\colon W_1 \times \cdots \times W_m \longrightarrow W$, which is explicitly defined by $\psi(w_1, \dots, w_m) = \varphi(T_1(w_1), \dots, T_m(w_m))$, is multilinear.

(b) The map $\psi' := T \circ \varphi\colon V_1 \times \cdots \times V_m \longrightarrow V$, which is explicitly defined by $\psi'(v_1, \dots, v_m) = T(\varphi(v_1, \dots, v_m))$, is multilinear.

Problem 2. Let $\varphi\colon V_1 \times \cdots \times V_m \longrightarrow P$ be a tensor map, and let $\psi\colon V_1 \times \cdots \times V_m \longrightarrow W$ be a multilinear map.

(a) Prove that ψ is a tensor map if and only if there exists a linear map $T\colon W \longrightarrow P$ with $\varphi = T \circ \psi$.

(b) Suppose that ψ is also a tensor map. Show that the map T from (a)

- is unique if and only if $\langle \operatorname{im} \psi \rangle = W$;
- can be chosen to be a bijection if and only if $\dim(P) = \dim(W)$ (but not necessarily $\langle \operatorname{im} \psi \rangle = W$).

The definitions necessary for the last problem can be found at the beginning of Chapter 1.2 of the notes. (No results of Chapter 1.2 are needed.)

Problem 3. (a) Let $u, u' \in U$, $v, v' \in V$ and $\lambda \in K$. Show that $\lambda(u \otimes v) = (\lambda u) \otimes v = u \otimes (\lambda v)$, that $(u + u') \otimes v = u \otimes v + u' \otimes v$, and that $u \otimes (v + v') = u \otimes v + u \otimes v'$. (*Note*: this is trivial if you understand the definitions.)

(b) Let $u \otimes v \in U \otimes V$. Show that if $u = 0$ or $v = 0$, then $u \otimes v = 0$.

(c) Let $\{e_1, \dots, e_k\}$ be a basis of U and $\{f_1, \dots, f_l\}$ be a basis of V. Show that

$$\{\, e_i \otimes f_j : i \in [k], j \in [l] \,\}$$

is a basis of $U \otimes V$.

(*Hint for (c)*: You have to show that the given vectors are linearly independent. Do so by finding suitable functionals using the universal factorization property)