DAS UE – WS 2022/23

Make-up exam – 10.3.2023

Name: **Matrikelnr.:**

Write your name on every sheet!

There are **5 points** per problem, for **20 points** altogether. For a positive grade you need at least **10 points**. Good luck!

Problem 1. Let V be a finite-dimensional vector space over a field K. Recall that the multiplication on the exterior algebra $\bigwedge V = \bigoplus_{r\geq 0} \bigwedge^r V$ satisfies

$$(v_1 \wedge \cdots \wedge v_r) \wedge (w_1 \wedge \cdots \wedge w_s) = v_1 \wedge \cdots \wedge v_r \wedge w_1 \wedge \cdots \wedge w_s.$$

for v_i, $w_j \in V$. The multiplication is extended K-linearly to all of $\bigwedge V$.
Show: if $\alpha \in \bigoplus_{r\geq 0} \bigwedge^{2r+1} V$, that is, α is a sum of simple alternating tensors of *odd* degree, then $\alpha \wedge \alpha = 0 \in \bigwedge V$.

Problem 2. Let R be a ring and I a left ideal of R. Let N be a left R-module. Show that the map

$$\Phi\colon \begin{cases} \operatorname{Hom}_R(R/I, N) & \to \{\, n \in N : \forall a \in I : an = 0 \,\} \\ f & \mapsto f(1 + I) \end{cases}$$

is an isomorphism of abelian groups. (You don't need to check that the stated sets are abelian groups.)

Problem 3. Let

$$f(z) = e^{z/2}\Big(1 - 3z\Big)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} a_n z^n.$$

Determine the asymptotic growth of the sequence $(a_n)_{n\geq 0}$.

Problem 4. Let A be a finite set with subsets $A_1, \ldots, A_n$, and let $d_1, \ldots, d_n \geq 1$. Show: If

$$\Big|\bigcup_{i\in I} A_i\Big| \geq \sum_{i\in I} d_i$$

for all $I \subseteq [n]$, then there exist pairwise disjoint subsets $D_k \subseteq A_k$ with $|D_k| = d_k$.

Hint: Construct a bipartite graph in which A is one side, and the other side consists of a suitable number of copies of the sets A_k. Define the edge set of the graph so that the desired result can be derived from Hall's theorem.