Exercise Sheet 10
Due 10.12.2020

Problem 2. Let $V_1, \dots, V_m, W_1, \dots, W_m$ be vector spaces. Show that there is an isomorphism

$$\bigotimes_{i=1}^m \operatorname{Hom}(V_i, W_i) \to \operatorname{Hom}\Big(\bigotimes_{i=1}^m V_i, \bigotimes_{i=1}^m W_i\Big), \quad T_1 \otimes \cdots \otimes T_m \mapsto T_1 \otimes \cdots \otimes T_m.$$

Careful: The notation $T_1 \otimes \cdots \otimes T_m$ is overloaded and has a different definition on the left and the right side above! This isomorphism provides a justification to use the same notation.

Surjectivity of the map. For each V_i fix a basis $e_{i,1}, \dots, e_{i,r_i}$. We know that

$$\{\, e_{1,j_1} \otimes \cdots \otimes e_{m,j_m} : j_1 \in [r_1], \dots, j_m \in [r_m] \,\}$$

is a basis for $V_1 \otimes \cdots \otimes V_m$. As in the lecture, we set $\Gamma \coloneqq \Gamma(r_1, \dots, r_n) \coloneqq [r_1] \times \cdots \times [r_m]$. For $\gamma \in \Gamma$ define $e_\gamma \coloneqq e_{1,\gamma(1)} \otimes \cdots \otimes e_{m,\gamma(m)}$. Then $\{\, e_\gamma : \gamma \in \Gamma \,\}$ is a basis for $V_1 \otimes \cdots \otimes V_m$.

Fixing for each W_i a basis $f_{i,1}, \dots, f_{i,s_i}$, setting $\Delta = \Gamma(s_1, \dots, s_m)$, and making analogous definitions, we have that $\{\, f_\gamma : \gamma \in \Delta \,\}$ is a basis for $W_1 \otimes \cdots \otimes W_m$.

Let $T \in \operatorname{Hom}\Big(\bigotimes_{i=1}^m V_i, \bigotimes_{i=1}^m W_i\Big)$. Since T is linear, it is uniquely determined by its action on a basis of $V_1 \otimes \cdots \otimes V_m$. Each $T(e_\gamma)$ is an element of $W_1 \otimes \cdots \otimes W_m$, so it is a linear combination of f_δ's with $\delta \in \Delta$. Explicitly, for every $\gamma \in \Gamma$ and every $\delta \in \Delta$, there exist $c_{\gamma,\delta} \in K$ such that

$$T(e_{1,\gamma(1)} \otimes \cdots \otimes e_{m,\gamma(m)}) = T(e_\gamma) = \sum_{\delta \in \Delta} c_{\gamma,\delta} f_\delta.$$

So far we have just done linear algebra (in a notationally complicated setting) and represented T with respect to a nice basis. Now, in a first approach we might like to find $T_i \colon V_i \to W_i$ such that $T(e_\gamma) = T_1(e_{1,\gamma(1)}) \otimes \cdots \otimes T_m(e_{m,\gamma(m)})$. However, it's not at all clear how to do this, since the right hand side does not, in general, decompose nicely into a product of m suitable terms. In fact, in general we can't expect T to be of the form $T_1 \otimes \cdots \otimes T_m$; we can only expect it to be a linear combination of such maps. So let's try something easier: Let's try to show that a nice generating set of $\operatorname{Hom}\Big(\bigotimes_{i=1}^m V_i, \bigotimes_{i=1}^m W_i\Big)$ is the image of the map. Then linearity will do the rest.

Claim: For every $\gamma \in \Gamma$ and $\delta \in \Delta$, there exist $S_{\gamma,\delta,i} \in \operatorname{Hom}(V_i, W_i)$ such that $(S_{\gamma,\delta,1} \otimes \cdots \otimes S_{\gamma,\delta,m})(e_\gamma) = f_\delta$ and $(S_{\gamma,\delta,1} \otimes \cdots \otimes S_{\gamma,\delta,m})(e_{\gamma'}) = 0$ for $\gamma' \in \Gamma$ with $\gamma' \neq \gamma$.

Proof of Claim: Expanding the notation, we want

$$S_{\gamma,\delta,1}(e_{1,\gamma(1)}) \otimes \cdots \otimes S_{\gamma,\delta,m}(e_{m,\gamma(m)}) = f_{1,\delta(1)} \otimes \cdots \otimes f_{m,\delta(m)}.$$

But now it is clear how to achieve this: Choose $S_{\gamma,\delta,i} \in \operatorname{Hom}(V_i, W_i)$ such that

$$S_{\gamma,\delta,i}(e_{i,j}) = \begin{cases} f_{i,\delta(i)} & \text{if } j = \gamma(i), \\ 0 & \text{if } j \neq \gamma(i). \end{cases}$$

□(Claim)

Writing Ψ for the homomorphism $\bigotimes_{i=1}^m \operatorname{Hom}(V_i, W_i) \to \operatorname{Hom}\left(\bigotimes_{i=1}^m V_i, \bigotimes_{i=1}^m W_i\right)$ and setting $S_{\gamma,\delta} = S_{\gamma,\delta,1} \otimes \cdots \otimes S_{\gamma,\delta,m} \in \bigotimes_{i=1}^m \operatorname{Hom}(V_i, W_i)$ where $\gamma \in \Gamma$, $\delta \in \Delta$, and the $S_{\gamma,\delta,i}$ are as in the claim, we have

$$\Psi(S_{\gamma,\delta})(e_{\gamma'}) = \begin{cases} f_\delta & \text{if } \gamma' = \gamma \\ 0 & \text{if } \gamma' \neq \gamma \end{cases}.$$

Define

$$T_0 \coloneqq \sum_{\gamma\in\Gamma} \sum_{\delta\in\Delta} c_{\gamma,\delta} S_{\gamma,\delta} \in \bigotimes_{i=1}^m \operatorname{Hom}(V_i, W_i).$$

Then, using linearity, for all $\gamma \in \Gamma$,

$$\Psi(T_0)(e_\gamma) = \sum_{\gamma'\in\Gamma} \sum_{\delta\in\Delta} c_{\gamma',\delta} \Psi(S_{\gamma',\delta})(e_\gamma) = \sum_{\delta\in\Delta} c_{\gamma,\delta} \Psi(S_{\gamma,\delta})(e_\gamma) = \sum_{\delta\in\Delta} c_{\gamma,\delta} f_\delta.$$

Thus $\Psi(T_0) = T$.