Final Exam for Problem Session (UE)
28.1.2021

Problem 1. A sequence $(a_n)_{n\geq 0}$ in $\mathbb{R}$ satisfies

$$6a_n = a_{n-2} + a_{n-1} \quad (n \geq 2), \qquad a_0 = -1, \qquad a_1 = -1.$$

Determine the asymptotic growth of $(a_n)_{n\geq 0}$.

Problem 2. Let A be a finite set with subsets $A_1, \ldots, A_n$, and let $d_1, \ldots, d_n \geq 1$. Show: If

$$\Big|\bigcup_{i\in I} A_i\Big| \geq \sum_{i\in I} d_i$$

for all $I \subseteq [n]$, then there exist pairwise disjoint subsets $D_k \subseteq A_k$ with $|D_k| = d_k$.

Hint: Construct a bipartite graph in which A is one side, and the other side consists of a suitable number of copies of the sets A_k. Define the edge set of the graph so that the desired result can be derived from Hall's theorem.

Problem 3. Let V be a finite-dimensional vector space over a field K. Recall that the multiplication on the exterior algebra $\bigwedge V = \bigoplus_{r\geq 0} \bigwedge^r V$ satisfies

$$(v_1 \wedge \cdots \wedge v_r) \wedge (w_1 \wedge \cdots \wedge w_s) = v_1 \wedge \cdots \wedge v_r \wedge w_1 \wedge \cdots \wedge w_s.$$

for $v_i, w_j \in V$. The multiplication is extended K-linearly to all of $\bigwedge V$.

Show: if $\alpha \in \bigoplus_{r\geq 0} \bigwedge^{2r+1} V$, that is, α is a sum of simple alternating tensors of *odd* degree, then $\alpha \wedge \alpha = 0 \in \bigwedge V$.

Problem 4. Let R be a ring and consider the following commutative diagram of R-modules and R-morphisms.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{\varepsilon_1} & B_1 & \xrightarrow{\pi_1} & C_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \gamma_1} & & \\
0 & \longrightarrow & A_2 & \xrightarrow{\varepsilon_2} & B_2 & \xrightarrow{\pi_2} & C_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \gamma_2} & & \\
0 & \longrightarrow & A_3 & \xrightarrow{\varepsilon_3} & B_3 & \xrightarrow{\pi_3} & C_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Suppose that the columns are exact and the bottom two rows are exact. Show that the top row is exact at B_1.