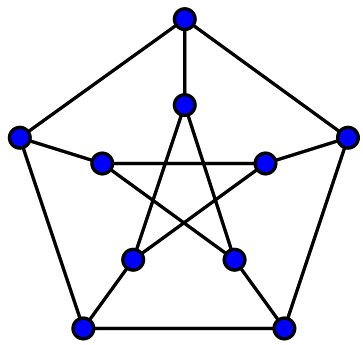
Exercise Sheet 6
Due 12.11.2020

Problem 1. Let $G = (V, E)$ be a plane graph and suppose that there exists a number $r \geq 3$ such that every face of G has at least r edges on its boundary. Prove that

$$|E| \leq \frac{r}{r-2}(|V| - 2),$$

and deduce from this that neither $K_{3,3}$ nor K_5 are planar.

Prove further that the Petersen graph P (see picture) is non-planar in two different ways: (a) by showing $K_{3,3}$ is a topological minor of P; (b) by showing K_5 is a minor of P.



(*Image Source: Wikipedia EN*)

Problem 2. Let $G = (V, E)$ be a graph. Show that the following statements are equivalent.

(i) G is a tree.

(ii) For every two distinct vertices $v, w \in V$ there is a unique v–w path in G.

(iii) G is minimally connected.

(iv) G is maximally acyclic.

(v) G is connected and $|E| = |V| - 1$.

(vi) G is acyclic and $|E| = |V| - 1$.

Problem 3. Let T be a tree on the vertex set $[n]$ (with $n \geq 2$), and let $s = (s_1, \ldots, s_{n-2})$ be its Prüfer code.

(a) Show that for each $v \in [n]$, the number of occurrences of v in s is $\deg(v) - 1$. In other words, show $|\{i \in [n-2] : s_i = v\}| = \deg(v) - 1$.

(b) Characterize all trees for which $s_1 = \cdots = s_{n-2}$.

(c) Characterize all trees for which $s_1, \ldots, s_{n-2}$ are pairwise distinct.