Exercise Sheet 2

Due 15.10.2020

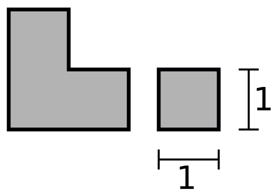
Problem 1. Let $A(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[\![z]\!]$ be a formal power series. Show that $A(z)$ has a reciprocal if and only if $a_0 \neq 0$, and that in this case the reciprocal is unique.

Supplementary question: What is the analogous condition when $A(z)$ is a power series over an arbitrary commutative (unital) ring $\mathbb{K}$?

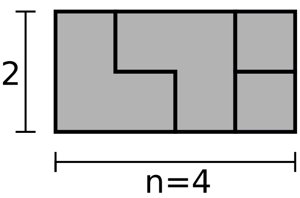
Problem 2. Use the identity $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \in \mathbb{C}[\![z]\!]$, and basic operations for power series (sum, product, differentation, integration) to find closed form expressions for the following power series.

$$\text{(i)} \sum_{n\geq 1} n^2 z^n \qquad \text{(ii)} \sum_{n\geq 0} \frac{n}{n+1} z^n \qquad \text{(iii)} \sum_{n\geq 0} \Big(\sum_{k=1}^{n} \frac{1}{k} \Big) z^n.$$

Problem 3. How many ways are there to fill completely without overlap a $2 \times n$ rectangle with pieces of the following types? The sides of the pieces are 1 and 2; the pieces can be rotated by a multiple of a right angle.



An example for $n = 4$ is shown below.



Let a_n denote the number of such tilings of the $2 \times n$ rectangle.

(a) Find a recursion for the sequence $(a_n)_{n\geq 1}$ and find a closed form expression for its (ordinary) generating function.

(b) Determine the asymptotic growth of $(a_n)_{n\geq 1}$.

Remark. If you find Problem 3 too hard, instead use OGFs to find a closed expression for the sequence $(a_n)_{n\geq 0}$ defined by the linear recurrence

$$a_{n+2} = 3a_n - 2a_{n+1} \quad (n \geq 0), \qquad a_0 = 2, a_1 = 3.$$