Exercise Sheet 13

Due 21.1.2021

Rings are assumed to be unital; ring homomorphisms are assumed to preserve the multiplicative identity. Problems 3 and 4 make use of some material from week 13 (cf. Section 5.3 of the Lecture Notes)

Problem 1. Let R be a ring and let M be an R-module. Show that a map $f\colon R^n \to R$ is an R-morphism if and only if there exist $m_1, \ldots, m_n \in M$ such that

$$f(r_1, \ldots, r_n) = r_1 m_1 + \cdots + r_n m_n \qquad \text{for all } r_1, \ldots, r_n \in R.$$

Conclude that M is finitely generated if and only if there exists an R-epimorphism $R^n \to M$ for some $n \geq 0$.

Problem 2. Let R be a ring and let $f\colon M \to N$ be an R-morphism. Suppose that $\ker(f)$ and $\operatorname{im}(f)$ are finitely generated. Show that M is finitely generated.

Problem 3. Consider the following commutative diagram of R-modules and R-morphisms, in which rows and columns are exact.

$$\begin{array}{ccccccc}
A & \xrightarrow{a} & B & \xrightarrow{b} & C & \longrightarrow & 0 \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\
D & \xrightarrow{d} & E & \xrightarrow{e} & F & \longrightarrow & 0 \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
L & \xrightarrow{l} & M & \xrightarrow{m} & N & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

Prove $\ker(h \circ e) = \operatorname{im} v + \operatorname{im} d$.

Problem 4. If

$$0 \longrightarrow B \xrightarrow{f} E \xrightarrow{g} A \longrightarrow 0$$

is a short exact sequence of R-modules, the triple (f, E, g) is an *extension of A by B.*

(a) Prove that, for any two R-modules A, B, at least one extension of A by B exists.

(b) Two extensions (f_1, E_1, g_1) and (f_2, E_2, g_2) of A and B are *equivalent* if there exists an R-morphism $h\colon E_1 \to E_2$ such that $h \circ f_1 = f_2$ and $g_2 \circ h = g_1$. Prove that such an R-morphism h is an isomorphism.

(c) Show that the following two are non-equivalent short exact sequences

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_4 \longrightarrow 0$$
$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_8 \longrightarrow \mathbb{Z}_4 \longrightarrow 0,$$

i.e., that these are extensions of $\mathbb{Z}_4$ by $\mathbb{Z}_2$ that are not equivalent.