Exercise Sheet 12

Due 14.1.2021

On this sheet all rings are assumed to be unital; every ring homomorphism is assumed to preserve the multiplicative identity.

Problem 1. Let $R = (R, +, \cdot)$ be a ring. For $a, b \in R$ define $a \cdot_{\mathrm{op}} b := b \cdot a$. Show that

(a) $R^{\mathrm{op}} := (R, +, \cdot_{\mathrm{op}})$ is a ring. (It is called the *opposite ring* of R.)

(b) Every left R-module M is a right R^{op}-module with $mr := rm$ for $r \in R$, $m \in M$.

(c) If R is commutative, then trivially $R \cong R^{\mathrm{op}}$. Is the converse true?

Problem 2. Let M be an abelian group and let $\operatorname{End} M$ be the set of all endomorphisms on M, that is, the set of all group homomorphisms $f : M \to M$. Show that

(a) $(\operatorname{End} M, +, \circ)$ is a ring, where $(f + g)(x) := f(x) + g(x)$ for $f, g \in \operatorname{End}(M)$ and $x \in M$.

(b) if R is a ring and $\mu : R \to \operatorname{End} M$ a ring homomorphism, then M is an R-module under the action $R \times M \to M$ given by $(\lambda, m) \mapsto \lambda m = (\mu(\lambda))(m)$.

(c) if R is a ring and M is an R-module, for every $\lambda \in R$, the map $\mu_\lambda \colon M \to M, m \mapsto \lambda m$ is a group endomorphism of M, and $\mu \colon R \to \operatorname{End}(M)$, $\lambda \mapsto \mu_\lambda$ is a ring homomorphism.

Problem 3. Let $T \colon V \to V$ be a linear endomorphism over a k-vector space V.

(a) Check that the vector space V can be made into a $k[x]$-module, defining multiplication as follows: if $f = \sum_{i=0}^{m} c_i x^i \in k[x]$ and $v \in V$, then

$$f \cdot v := \sum_{i=0}^{m} c_i T^i(v),$$

where T^i denotes the i-fold composition (that is $T^0 = \mathrm{id}_v$ and $T^i := T^{i-1} \circ T$ for $i \geq 1$).

(b) Show that $W \subseteq V$ is a $k[x]$-submodule of V if and only if W is a k-vector subspace of V and $T(W) \subseteq W$.

Problem 4. Let M be an R-module. If S is a non-empty subset of M, define the *annihilator of S in R* to be

$$\operatorname{Ann}_R S = \{\, r \in R \mid \forall x \in S : \; rx = 0_M \,\}.$$

(a) Show that $\operatorname{Ann}_R S$ is a left ideal of R and that it is a two-sided ideal whenever S is a submodule of M.

(b) Let M be an R-module. If $r, s \in R$ show that

$$r - s \in \operatorname{Ann}_R M \Longrightarrow \forall x \in M : \; rx = sx.$$

Deduce that M can be considered as an $R/\operatorname{Ann}_R M$-module. Show that the annihilator of M in $R/\operatorname{Ann}_R M$ is zero.