Exercise Sheet 11

Due 17.12.2020

Problem 1. Identify $\bigwedge^2(\mathbb{R}^3)$ with $\mathbb{R}^3$ by identifying $e_1 \wedge e_2$ with e_3, $e_2 \wedge e_3$ with e_1 and $e_3 \wedge e_1$ with e_2. Show that under this identification, the exterior product $v \wedge w \in \bigwedge^2(\mathbb{R}^3) = \mathbb{R}^3$ is the same as the cross product $v \times w \in \mathbb{R}^3$.

Problem 2. Let K be a field and let $A = (a_{i,j})_{i,j\in[n]}$ be an $n \times n$-matrix with entries in K. Denote by $e_1, \dots, e_n$ the standard basis of K^n. Show that

$$(Ae_1) \wedge \cdots \wedge (Ae_n) = \sum_{\pi \in \mathcal{S}_n} \operatorname{sgn}(\pi) a_{1,\pi(1)} \cdots a_{n,\pi(n)}\, e_1 \wedge \cdots \wedge e_n.$$

Problem 3. Let V be a (finite-dimensional) vector space over a field K.

(a) $v_1, \dots, v_k \in V$ are linearly independent if and only if $v_1 \wedge \cdots \wedge v_k \neq 0$ (in $\bigwedge^k V$).

(b) Let $v_1, \dots, v_k \in V$ and $w_1, \dots, w_k \in V$ be two tuples of linearly independent vectors. Then $\langle v_1, \dots, v_k \rangle = \langle w_1, \dots, w_k \rangle$ if and only if there exists $c \in K^\times$ such that

$$v_1 \wedge \cdots \wedge v_k = c w_1 \wedge \cdots \wedge w_k.$$

(Thus k-dimensional subspaces of V are in bijection with *pure* tensors in $\wedge^k V$ up to scalar multiples.)