Exercise Sheet 10
Due 10.12.2020

Throughout, vector spaces are considered over an arbitrary but fixed field K and assumed to be finite-dimensional.

Problem 1. Let U, V be vector spaces.

(a) Let $u_1, \dots, u_n \in U$ be linearly independent. For $x \in U \otimes V$ having a representation $x = \sum_{i=1}^n u_i \otimes v_i$ with $v_1, \dots, v_n \in V$, show that $v_1, \dots, v_n \in V$ are uniquely determined.

(b) Show that there exists an isomorphism of vector spaces $\varphi\colon U \otimes V^* \to \operatorname{Hom}(V, U)$ satisfying $\varphi(u \otimes f)(v) = f(v)u$ for all $u \in U$, $f \in V^*$, and $v \in V$. [1]

Problem 2. Let $V_1, \dots, V_m, W_1, \dots, W_m$ be vector spaces. Show that there is an isomorphism

$$\bigotimes_{i=1}^m \operatorname{Hom}(V_i, W_i) \to \operatorname{Hom}\Big(\bigotimes_{i=1}^m V_i, \bigotimes_{i=1}^m W_i\Big), \quad T_1 \otimes \cdots \otimes T_m \mapsto T_1 \otimes \cdots \otimes T_m.$$

Careful: The notation $T_1 \otimes \cdots \otimes T_m$ is overloaded and has a different definition on the left and the right side above! This isomorphism provides a justification to use the same notation.[2]

Problem 3. Let $(V_i)_{i\in I}$ be a family of vector spaces with a finite index set I. Recall that the (*external*) *direct sum* is

$$\bigoplus_{i\in I} V_i = \{\, (v_i)_{i\in I} : v_i \in V_i \,\}$$

with the componentwise vector space structure. For every $k \in I$, there is a monomorphism $\varepsilon_k\colon V_k \to \bigoplus_{i\in I} V_i,\ v \mapsto (0, \dots, 0, v, 0, \dots, 0)$ whose only nonzero entry is in the k-th component.

(a) Show that $\bigoplus_{i\in I} V_i$ (together with $(\varepsilon_i)_{i\in I}$) satisfies the following universal property: If W is a vector space and $(f_i \in \operatorname{Hom}(V_i, W))_{i\in I}$ are homomorphisms, then there exists a unique homomorphism $g\colon \bigoplus_{i\in I} V_i \to W$ such that $g \circ \varepsilon_i = f_i$ for all $i \in I$.

[1] The map φ is injective even for infinite-dimensional vector spaces, but its image consists of those linear maps $g \in \operatorname{Hom}(V, U)$ having finite-dimensional image (i.e., $\dim g(V) < \infty$).

[2] This isomorphism depends crucially on V_i, W_i being finite-dimensional vector spaces. For a tensor product of modules, it need not be injective nor surjective.

(b) Let $(U_j)_{j\in J}$ be another family of vector spaces. Show that there is a canonical isomorphism[3]

$$\Big(\bigoplus_{j\in J} U_j\Big) \otimes \Big(\bigoplus_{i\in I} V_i\Big) \cong \bigoplus_{i\in I, j\in J} (U_j \otimes V_i).$$

[3]This one actually works also for infinite-dimensional vector spaces and arbitrary index sets I, J. The external direct-sum then only consists of those tuples where all but finitely many components are zero.