Exercise 1. One has $\mathbb{Z}_p/p^i\mathbb{Z}_p \cong \mathbb{Z}/p^i\mathbb{Z}$ for all $i \geq 0$. (Analyze $\ker \pi_i : \mathbb{Z}_p \to \mathbb{Z}/p^i\mathbb{Z}$.)

Exercise 2. If $\alpha, \beta \in \mathbb{Z}_p$ with $|\alpha|_p \neq |\beta_p|$, then $|\alpha + \beta|_p = \max\left\{|\alpha|_p, |\beta|_p\right\}$.

Exercise 3. The ring $\mathbb{Z}_p$ is a domain (meaning, 0 is the only zero-divisor) and the invertible elements are

$$\mathbb{Z}_p^\times = \left\{\alpha \in \mathbb{Z}_p : \pi_1(\alpha) \neq 0\right\} = \left\{\sum_{n=0}^{\infty} a_i p^i : 0 \leq a_i \leq p-1,\ a_0 \neq 0\right\}.$$

Exercise 4. Show Legendre's formula: if p is a prime and $n \in \mathbb{N}_0$, then

$$\mathrm{v}_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor = \frac{n - \sigma_p(n)}{p-1}.$$

(note that the sum is actually finite). Here,

- $\sigma_p(n)$ is the sum of digits of n in its base p representation, e.g., $\sigma_3(10) = \sigma_3(1 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0) = 1 + 1 = 2$.
- for all $x \in \mathbb{R}$, the expression $\lfloor x \rfloor = \max\{\, m \in \mathbb{Z} : m \leq x \,\}$ denotes the floor function.

Exercise 5. In the p-adic topology, determine the convergence radii of the power series

$$\exp(x) \coloneqq \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{and} \qquad \log(1+x) \coloneqq \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n.$$