PRIMES AND ABSOLUTELY OR NON-ABSOLUTELY IRREDUCIBLE ELEMENTS IN ATOMIC DOMAINS

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Abstract. We give examples of atomic integral domains satisfying each of the eight logically possible combinations of existence or non-existence of the following kinds of elements: 1) primes, 2) absolutely irreducible elements that are not prime, and 3) irreducible elements that are not absolutely irreducible. A non-zero non-unit is called absolutely irreducible (or, a strong atom) if every one of its powers factors uniquely into irreducibles.

1. Introduction

Among atomic domains, that is, domains in which every non-zero non-unit is a product of irreducibles, unique factorization domains are characterized by the fact that all irreducibles are prime.

Chapman and Krause [10] showed for rings of integers in number fields that $\mathcal{O}_K$ is a UFD if and only if every irreducible element is absolutely irreducible — meaning that each of its powers factors uniquely into irreducibles — a weaker property than prime.

Their result prompts the question whether this characterization of unique factorization domains holds in greater generality: among Dedekind domains, for instance. The answer is no.

We have the following implications (whose converses do not hold):

$$\text{prime} \quad\Longrightarrow\quad \text{absolutely irreducible} \quad\Longrightarrow\quad \text{irreducible}$$

This gives us three kinds of elements that may or may not exist in a given domain:

(i) Primes
(ii) Absolutey irreducibles that are not prime
(iii) Irreducibles that are not absolutely irreducible

— and, therefore, eight logically possible combinations of existence or non-existence of each kind of element.

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Monoid examples for all eight scenarios are easy to find, compared to examples in integral domains. For instance, Baginski and Kravitz [7] provide examples of non-factorial monoids whose irreducibles are all absolutely irreducible.

We will show that all eight scenarios occur in atomic domains. Some cases are trivial: Atomic domains without any irreducible elements, let alone absolutely irreducible or prime elements, are just fields. (There are also non-atomic domains without irreducible elements, the so-called antimatter domains [2].)

Atomic domains with primes and no other irreducible elements are just UFDs (that are not fields), as mentioned. For the remaining six non-trivial combinations we now proceed to give examples.

In the following table, plus indicates existence, and minus, non-existence. R_1 and R_2 are certain Dedekind domains with class group $\mathbb{Z}^n$, see Proposition 7.1 and Proposition 8.1, respectively.

Section	Example	Irreducible but not absolutely irreducible	Absolutely irreducible but not prime	Prime
Section 6	$\mathbb{Z}[\sqrt{-14}]$	+	+	+
Section 4	$\mathrm{Int}(\mathcal{O}_K)$	+	+	-
Section 5	$\mathbb{R}+X\mathbb{C}[X]$	+	-	+
Section 3	$\mathbb{R}+X\mathbb{C}[[X]]$	+	-	-
Section 8	R_2	-	+	+
Section 7	R_1	-	+	-
Section 1	UFDs	-	-	+
Section 1	Fields	-	-	-

2. Preliminaries

2.1. **Factorization terms.** In this section, we recall some concepts related to factorization. For a comprehensive introduction to non-unique factorizations, we refer to the textbook by Geroldinger and Halter-Koch [15].

The terms that we here define for a monoid H we will use mostly (but not only) in the context of an integral domain R. In that case, the monoid in question is understood to be $(R \setminus \{0\}, \cdot)$.

Definition 2.1. Let $(H, \cdot)$ be a cancellative commutative monoid.

(i) $r \in H$ is said to be *irreducible* in H (or, an *atom* of H) if it is a non-unit that is not a product of two non-units of H.

(ii) A *factorization* of $r \in H$ is an expression

$$r = a_1 \cdots a_n,$$

where $n \geq 1$ and a_i is irreducible in H for $1 \leq i \leq n$.

(iii) $r, s \in H$ are *associated* in H if there exists a unit $u \in H$ such that $r = us$. We denote this by $r \sim s$.
(iv) Two factorizations into irreducibles of the same element,
$$r = a_1 \cdots a_n = b_1 \cdots b_m, \tag{1}$$
are called *essentially the same* if $n = m$ and, after re-indexing, $a_j \sim b_j$ for $1 \leq j \leq m$. Otherwise, the factorizations in (1) are called *essentially different*.
(v) $(H, \cdot)$ is called *atomic* if every non-unit has a factorization.
(vi) $(H, \cdot)$ is *factorial* if H is atomic and any two factorizations of an element are essentially the same.
(vii) $(H, \cdot)$ is *half-factorial* if H is atomic and any two factorizations of an element have the same number of irreducible factors.

Definition 2.2. Let H be a cancellative commutative monoid.
(i) An irreducible element $r \in H$ is called *absolutely irreducible* (or, a *strong atom*), if for all natural numbers n, every factorization of r^n is essentially the same as $r^n = r \cdots r$.
(ii) If $r \in H$ is irreducible, but not absolutely irreducible, it is called *non-absolutely irreducible*.

2.2. **Transfer homomorphisms.** Transfer homomorphisms are a key tool in factorization theory. They are used to study non-unique factorization in a domain (or monoid) using a simpler "model" monoid. In this section, we show that transfer homomorphism preserve absolute irreducibility (in the forward direction).

Definition 2.3. [15, Definition 3.2.1] Let H and M be commutative monoids. A homomorphism $\theta\colon H \longrightarrow M$ is called a transfer homomorphism if it has the following properties:
(i) $M = \theta(H)M^\times$ and $\theta^{-1}(M^\times) = H^\times$
(ii) If $u \in H$ and $b,\, c \in M$ such that $\theta(u) = bc$, then there exists $v,\, w \in H$ such that $u = vw$ and $\theta(v) \sim b$ and $\theta(w) \sim c$.

Here and elsewhere this paper, the group of units of the monoid H is denoted by $H^\times$.

Fact 2.4. [15, Proposition 3.2.3] *Let $\theta\colon H \longrightarrow M$ be a transfer homomorphism.*
(i) *An element $u \in H$ is irreducible in H if and only if $\theta(u)$ is irreducible in M.*
(ii) *H is atomic if and only if M is atomic.*

Lemma 2.5. *Let $\theta\colon H \longrightarrow M$ be a transfer homomorphism, and $c \in H$ an irreducible element. If c is absolutely irreducible in H, then $\theta(c)$ is absolutely irreducible in M.*

Proof. If $\theta(c)$ is not absolutely irreducible in M, then there exists an irreducible element $\theta(b)$ of M, not associated to $\theta(c)$, that divides $\theta(c)^m$

for some $m \in \mathbb{N}$. It follows that b is an irreducible element of H, not associated to c, which divides c^m. $\square$

Remark 2.6. The converse of Lemma 2.5 does not hold. A non-absolutely irreducible element of H may be mapped to an absolutely irreducible element of M by a transfer homomorphism $\theta\colon H \longrightarrow M$. We illustrate this by an example: If H is a commutative monoid and $M = (\mathbb{N}_0, +)$, then the function

$$\begin{aligned} \theta\colon H &\longrightarrow M \\ a &\longmapsto \ell(a) \end{aligned}$$

is a transfer homomorphism.

If $c \in H$ is irreducible in H then $\theta(c)$ is irreducible in M, by Fact 2.4. The unique irreducible of M is 1 (which is prime). Hence $\theta(c) = 1$ for all irreducible $c \in H$, and in particular, every irreducible in M is absolutely irreducible. However, there exist half-factorial monoids H that contain non-absolutely irreducibles, for instance $H = \mathcal{O}_K \setminus \{0\}$ with $\mathcal{O}_K$ a ring of algebraic integers whose class group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (see Fact 6.2 below).

2.3. **Krull monoids.** A cancellative commutative monoid is a *Krull monoid* if it is v-noetherian and completely integrally closed. A *Krull domain* is an integral domain D such that $D \setminus \{0\}$ is a Krull monoid. We refer to [15, Chapter 2] for the algebraic theory of Krull monoids.

Let H be a Krull monoid and let $\mathfrak{X}(H)$ denote the set of nonempty divisorial prime (semigroup) ideals. The nonempty divisorial ideals of H form a free abelian monoid with basis $\mathfrak{X}(H)$ with respect to the divisorial product; the divisorial fractional ideals form a free abelian group on the same basis. Explicitly, every nonempty divisorial ideal $\mathfrak{a}$ of H is uniquely expressible as a divisorial product of prime ideals

$$\mathfrak{a} = \mathfrak{p}_1 \cdot \cdots _v \, \mathfrak{p}_r = \Big(\prod_{\mathfrak{p} \in \mathfrak{X}(H)} \mathfrak{p}^{\mathsf{v}_\mathfrak{p}(\mathfrak{a})} \Big)_v .$$

The set $\{\, \mathfrak{p} \in \mathfrak{X}(H) : \mathsf{v}_\mathfrak{p}(\mathfrak{a}) > 0 \,\} = \{\, \mathfrak{p} \in \mathfrak{X}(H) : \mathfrak{p} \supseteq \mathfrak{a} \,\}$ is the *support* of $\mathfrak{a}$. The class group $G = \mathcal{C}(H)$ of H is the quotient of the group of divisorial fractional ideals of H by the subgroup of principal fractional ideals. Let $[\mathfrak{a}] \in G$ denote the class of $\mathfrak{a}$. In factorization theory, the set $G_0 = \{\, [\mathfrak{p}] : \mathfrak{p} \in \mathfrak{X}(H) \,\}$ of classes containing prime divisors is of central importance.

We have two main examples of Krull monoids in mind: the first are the Dedekind domains, which are Krull domains (in fact, they are precisely the one-dimensional Krull domains). In a Dedekind domain D, the map $\mathfrak{a} \mapsto \mathfrak{a} \setminus \{0\}$ is a bijection between ring ideals of D and divisorial semigroup ideals of $D \setminus \{0\}$. The divisorial product is the usual product of ideals, and the class group is the usual one (the group of fractional ideals by principal fractional ideals).

The following is another class of Krull monoids.

Definition 2.7. Let G be an additively written abelian group and $G_0 \subseteq G$ a nonempty subset. Let $\mathcal{F}(G_0)$ be the free abelian monoid with basis G_0.

(i) The elements of $\mathcal{F}(G_0)$ are called *sequences* over G_0 and are of the form

$$S = \prod_{g \in G_0} g^{n_g},$$

where $n_g = \mathsf{v}_g(S) \in \mathbb{N} \cup \{0\}$ with $n_g = 0$ for almost all $g \in G_0$.

(ii) The *length* of a sequence S is

$$|S| = \sum_{g \in G_0} \mathsf{v}_g(S) \quad \in \ \mathbb{N} \cup \{0\}$$

and the *sum* of S is

$$\sigma(S) = \sum_{g \in G_0} \mathsf{v}_g(S) g \quad \in \ G.$$

The *support* of S is

$$\operatorname{supp}(S) = \{\, g \in G_0 : \mathsf{v}_g(S) > 0 \,\}.$$

(iii) The monoid

$$\mathcal{B}(G_0) = \Big\{ S \in \mathcal{F}(G_0) \ : \ \sigma(S) = 0 \Big\}$$

is called the *monoid of zero-sum sequences* over G_0 or the *block monoid*.

The irreducibles of $\mathcal{B}(G_0)$ are the *minimal zero-sum sequences*: nonempty sequences whose sum is 0, but which do not contain a nonempty, proper zero-sum subsequence whose sum is 0.

The following key theorem links the factorization theory of Krull monoids to that of monoids of zero-sum sequences, showing that the latter serve as a combinatorial models for the factorization in Krull monoids and, in particular, Dedekind domains. A proof can be found in [15, Theorem 3.4.10.1]. For more expository accounts of this theory see [14, Theorem 1.3.4.2], the survey [16], or the expository article [6].

Theorem 1. *Let H be a Krull monoid with class group G, and let $G_0 \subseteq G$ be the set of classes containing prime divisors. Then there exists a transfer homomorphism*

$$\theta \colon H \to \mathcal{B}(G_0), \quad a \mapsto [\mathfrak{p}_1] \cdots [\mathfrak{p}_r],$$

where $aH = \mathfrak{p}_1 \cdot \cdot_v \, \mathfrak{p}_r$ with $\mathfrak{p}_i \in \mathfrak{X}(H)$.

The homomorphism θ is called the *block homomorphism* of H.

3. Rings whose irreducible elements are all non-absolutely irreducible

Examples of atomic domains all of whose irreducibles are non-absolutely irreducible occur among generalised power series rings.

Remark 3.1. Let $K_1 \subseteq K_2$ be fields, $n \in \mathbb{N}$, and $R = K_1 + X^n K_2[[X]]$. Let H be the multiplicative monoid $R \setminus \{0\}$ and M_n be the numerical monoid $\{0\} \cup (n + \mathbb{N}_0)$. Then the monoid homomorphism

$$\begin{aligned} \theta : H &\longrightarrow M_n \\ uX^\ell &\longmapsto \ell \end{aligned}$$

is a transfer homomorphism, where u is a unit of $K_2[[X]]$ and $\ell \in M_n$.

Proposition 3.2. *Let $K_1 \subseteq K_2$ be fields and $n \in \mathbb{N}$, and set $R = K_1 + X^n K_2[[X]]$. Then the following hold.*

(i) *R has absolutely irreducible elements if and only if $K_1 = K_2$ and $n = 1$.*
(ii) *R has prime elements if and only if $K_1 = K_2$ and $n = 1$.*

Proof. For (i), suppose $K_1 = K_2$ and $n = 1$. Then R is a unique factorization domain.

Conversely, suppose $K_1 = K_2$ and $n > 1$. Then the units of R are the elements of R with constant term in $K_2 \setminus \{0\}$. It follows from Remark 3.1 and Fact 2.4 that the irreducible elements of R are the polynomials of the form $r = uX^m$, where u is a unit of $K_2[[X]]$ and $n \leq m \leq 2n - 1$. Every irreducible of the form r is not absolutely irreducible since for some $n \leq t \leq 2n - 1$, with $t \neq m$,

$$r^t = u^t X^t \cdot \underbrace{X^t \cdots X^t}_{m-1 \text{ copies}}$$

is a factorization of r essentially different from

$$\underbrace{r \cdots r}_{t \text{ copies}}.$$

Suppose $K_1 \neq K_2$. Then the units of R are the elements of R with constant term in $K_1 \setminus \{0\}$. Similarly, the irreducible elements of R are the polynomials of the form uX^m, where u is a unit of $K_2[[X]]$ and $n \leq m \leq 2n - 1$. Moreover, $u_1 X^m \not\simeq u_2 X^m$ if u_1 is a unit in $K_1[[X]]$ and $u_2 \in K_2[[X]]$ has its constant term in $K_2 \setminus K_1$. It follows that each irreducible of the form uX^m is not absolutely irreducible since

$$(uX^m)^2 = ucX^m \cdot uc^{-1}X^m$$

is a factorization of $(uX^m)^2$ essentially different from $uX^m \cdot uX^m$, where $c \in K_2 \setminus K_1$.

For (ii), if $K_1 = K_2$ and $n = 1$, then $R = K_2[[X]]$ is a unique factorization domain, otherwise, the assertion follows from (i) and the fact that every prime element is absolutely irreducible. $\square$

4. Rings with both absolutely and non-absolutely irreducible elements, but no primes

As an example of an atomic domain that has no prime element and contains both absolutely and non-absolutely irreducible elements, we propose the ring of integer-valued polynomials $\mathrm{Int}(\mathbb{Z})$, or, more generally, $\mathrm{Int}(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers in a number field K. For a domain D with quotient field K, the ring of integer-valued polynomials on D is

$$\mathrm{Int}(D) = \{f \in K[x] \mid f(D) \subseteq D\}.$$

The fixed divisor of $f \in \mathrm{Int}(D)$, abbreviated $\mathrm{fd}(f)$, is the ideal of D generated by $f(D)$. If the fixed divisor is a principal ideal, we say $\mathrm{fd}(f) = c$, by abuse of notation, for $\mathrm{fd}(f) = (c)$. A polynomial $f \in \mathrm{Int}(D)$ with $\mathrm{fd}(f) = 1$ is called image-primitive. It is clear that $\mathrm{fd}(f) \cdot \mathrm{fd}(g) \supseteq \mathrm{fd}(fg)$, but note that the incusion can be strict. In any case, all divisors in $\mathrm{Int}(D)$ of an image-primitive polynomial $f \in \mathrm{Int}(D)$ are image-primitive.

4.1. **No prime elements.** Anderson, Cahen, Chapman, and Smith [3] showed that $\mathrm{Int}(\mathbb{Z})$ has no prime element by using the fact that $\mathrm{Int}(\mathbb{Z})$ is a Prüfer domain whose maximal ideals are known and are not principal. They argue that a Prüfer domain never has any principal prime ideals other than (0) and, possibly, maximal ideals. At the same time, no maximal ideal of $\mathrm{Int}(\mathbb{Z})$ is principal (or even finitely generated). Their argument readily generalizes to $\mathrm{Int}(\mathcal{O}_K)$.

We will here give an elementary, more explicit, proof that $\mathrm{Int}(\mathbb{Z})$ (and, more generally, $\mathrm{Int}(\mathcal{O}_K)$) has no prime element, by exhibiting, for every potential prime element p, a product ab such that p divides ab, but p divides neither a nor b.

The only non-trivial fact needed is that every non-constant polynomial in $\mathbb{Z}[x]$ has zeros modulo infinitely many primes, and, more generally, for every number field K, every non-constant polynomial in $\mathcal{O}_K[x]$ has zeros modulo infinitely many maximal ideals of $\mathcal{O}_K$.

To see this we refer to a few facts about d-rings, a notion introduced independently by Brizolis [8], and Gunji and McQuillan [19].

Definition 4.1. A domain D is a *d-ring* if for every non-constant polynomial $f \in D[x]$ there exists a maximal ideal M of D and an element $d \in D$ such that $f(d) \in M$.

So, D being a *d*-ring just means that a polynomial $f \in D[x]$ cannot map D into the set of units of D unless f is a constant. It is easy to see that $\mathbb{Z}$ is a d-ring. Indeed, any $f \in \mathbb{Z}[x]$ such that $f(\mathbb{Z}) \subseteq \{1, -1\}$ must be constant.

Alternatively, d-rings can be characterized as those domains for which every integer-valued rational function is an integer-valued polynomial. We summarize what we need to know about d-rings (cf. [9], §VII.2).

Fact 4.2. [8, Lemma 1.3], [19, Prop. 1]. *The following are equivalent:*

(i) *D is a d-ring.*
(ii) *For every non-constant $f \in D[x]$, the intersection of the maximal ideals M of D for which f has a zero modulo M is (0).*
(iii) *For every non-constant $f \in \mathrm{Int}(D)$, there exists a maximal ideal M of D and an element $d \in D$ such that $f(d) \in M$.*

We conclude from Fact 4.2 that every non-constant polynomial in $\mathbb{Z}[x]$ has zeros modulo infinitely many primes.

Lemma 4.3 (Anderson, Cahen, Chapman, and Smith [3])**.** $\mathrm{Int}(\mathbb{Z})$ *has no prime element.*

Proof. First, no constant can be prime, because, if $p \in \mathbb{Z}$ is a non-zero non-unit, then p divides $(x - r_1) \cdots (x - r_p)$, where $r_1, \ldots, r_p$ is a complete set of residues modulo $p\mathbb{Z}$, but, since $(x - r_i)/p$ is not integer-valued, p does not divide any individual linear factor.

Now consider $G \in \mathrm{Int}(\mathbb{Z})$ non-constant, $G(x) = \frac{g}{d}$ with $g(x) \in \mathbb{Z}[x]$ and $d \in \mathbb{Z}$. Let $p \in \mathbb{Z}$ be prime such that g has a zero modulo p but the polynomial function induced by g is not constant zero modulo p. (Such a prime p exists because g has a zero modulo infinitely many primes, but only finitely many primes divide the fixed divisor of g.)

Since $g \in \mathbb{Z}[x]$, the residue class of $g(r)$ modulo p depends only on the residue class of r modulo p. Let $r_1, \ldots, r_k$ be a complete set of representatives of those residue classes modulo p on which g takes a non-zero value modulo p. Let $h(x) = \prod_{i=1}^{k}(x - r_i)$.

Then, both $\frac{h(x)G(x)}{p}$ and $\frac{(G(x)+p)h(x)}{p}$ are integer-valued. This means that G divides $\frac{(G(x)+p)h(x)G(x)}{p}$, but G divides neither $G(x)+p$ (not even in $\mathbb{Q}[x]$) nor $\frac{h(x)G(x)}{p}$ (because $\frac{h(x)}{p}$ is not integer-valued). ☐

To generalize to $\mathrm{Int}(\mathcal{O}_K)$, we note that $\mathcal{O}_K$ is a d-ring for every number field K:

Fact 4.4. [19, Prop. 3, Corollary 2]. *Let $D \subseteq R$ be domains, and D a d-ring.*

(i) *If R is integral over D, then R is a d-ring.*
(ii) *If R is finitely generated as a ring over D, then R is a d-ring.*

Since $\mathbb{Z}$ is a d-ring, it follows by Fact 4.4 that $\mathcal{O}_K$, too, is a d-ring.

Lemma 4.5. *Let $\mathcal{O}_K$ be the ring of integers in a number field K. Then* $\mathrm{Int}(\mathcal{O}_K)$ *has no prime element.*

Proof. First, no constant can be prime, because, if $p \in \mathcal{O}_K$ is a non-zero non-unit, then p divides $(x - r_1) \cdots (x - r_k)$, where $r_1, \ldots, r_k$ is a complete set of residues modulo $p\mathcal{O}_K$, but p does not divide any individual linear factor.

Now consider $G \in \mathrm{Int}(\mathcal{O}_K)$ non-constant, $G(x) = \frac{g}{d}$ with $g(x) \in \mathcal{O}_K[x]$ and $d \in \mathcal{O}_K$. Because there are only finitely many ramified primes and only finitely many primes dividing the fixed divisor of g, there exists an unramified prime $p \in \mathbb{Z}$, say $p\mathcal{O}_K = P_1 \cdot \ldots \cdot P_r$, such that

(i) g has a zero modulo P_1,
(ii) no P_i divides the fixed divisor of g.

Let k be the maximal number of different residue classes modulo any one P_j on which g assumes a non-zero value modulo P_j. Choose $R = \{r_1, \ldots, r_k\} \subseteq \mathcal{O}_K$ such that

(i) for each P_j, R contains a complete set of representatives of the residue classes modulo P_j on which g assumes a non-zero value modulo P_j
(ii) $g(r_i)$ is not zero modulo P_1 for any $r_i \in R$.

Let $h = \prod_{i=1}^{k}(x - r_i)$, then

$$G \,\Big|\, \frac{g(x)h(x)(g(x)+dp)}{dp} = G(x)\frac{h(x)(g(x)+dp)}{p}, \text{ but}$$

$$G \nmid \frac{g(x)h(x)}{dp} \text{ and } G \nmid (g(x)+dp).$$

□

4.2. **Absolutely irreducible elements.** Examples of absolutely irreducible elements of $\mathrm{Int}(\mathbb{Z})$ include the binomial polynomials

$$\binom{x}{n} = \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!}$$

for $n > 1$. If n is prime, this is elementary, as already McClain [23] remarked in her honor's thesis. For general n, it is non-trivial and was shown by Rissner and the fifth author [26]. Their result has been generalized to function fields by Tichy and the fifth author [27], but we are here concerned with rings of integer-valued polynomials on number fields, where we can provide quite elementary examples of absolutely irreducible elements as follows:

Lemma 4.6. *For any number field K, there exist absolutely irreducible elements in* $\mathrm{Int}(\mathcal{O}_K)$.

Proof. Let $p \in \mathcal{O}_K$ be an irreducible element with square-free factorization into prime ideals; $p\mathcal{O}_K = P_1 \cdot \ldots \cdot P_r$. (Such an element exists because there are unramified primes and $\mathcal{O}_K$ is atomic.) Then let $q = \max_{1 \le j \le r}[\mathcal{O}_K : P_j]$. W.l.o.g., $[\mathcal{O}_K : P_1] = q$. Let $r_1, \ldots, r_q$ be a complete system of residues modulo P_1, containing one (but not more than one) complete system of residues modulo P_i for each i and not containing a complete system of residues modulo any other primes. Set

$$f(x) = (x - r_1)\cdots(x - r_q) \quad \text{and} \quad F(x) = \frac{f(x)}{p}.$$

Then F is absolutely irreducible.

To see this, suppose F^m factors as $F^m = c_1 g_1 \dots c_s g_s$ with $c_i \in K$, g_i monic in $K[x]$, and $c_i g_i \in \mathrm{Int}(\mathcal{O}_K)$ irreducible for all i. Note that F^m is image-primitive, and, therefore, so is $c_i g_i$ for all i, meaning $c_i = (\mathrm{fd}(g_i))^{-1}$.

Let v_k be the normalized essential valuation corresponding to P_k. Then $v_1(c_i) = -v_1(\mathrm{fd}(g_i))$, and $\sum_{i=1}^s v_1(c_i) = v_1(p^{-m}) = -m$. So, $\sum_{i=1}^s v_1(\mathrm{fd}(g_i)) = m$, which can only happen if each g_i is a power of f; $g_i = f^{m_i}$ with $\sum_{i=1}^s m_i = m$.

Now $c_i = (\mathrm{fd}(g_i))^{-1} = p^{-m_i}$, and, therefore, $c_i g_i = f^{m_i} p^{-m_i} = F^{m_i}$. As $c_i g_i$ was assumed irreducible, $m_i = 1$ follows. □

There are many other examples of absolutely irreducible elements in $\mathrm{Int}(\mathcal{O}_K)$, or more generally in $\mathrm{Int}(D)$, where D is a Dedekind domain with at least one finite residue field and torsion class group [12, Corollary 8.9.].

4.3. **Non-absolutely irreducible elements.** As an example of a non-absolutely irreducible element of $\mathrm{Int}(\mathbb{Z})$, consider

$$f = \frac{x(x^2+3)}{2}, \text{ noting that } f^2 = \frac{x^2(x^2+3)}{4} \cdot (x^2+3).$$

This example, taken from the third author's paper [24] on non-absolutely irreducible integer-valued polynomials, generalises to $\mathrm{Int}(\mathcal{O}_K)$ as follows.

Lemma 4.7. *For any number field K, there exist non-absolutely irreducible elements in* $\mathrm{Int}(\mathcal{O}_K)$*.*

Proof. Let $p \in \mathcal{O}_K$ be an irreducible element with square-free factorization into prime ideals; $p\mathcal{O}_K = P_1 \cdot \ldots \cdot P_r$. (Such an element exists because there are unramified primes and $\mathcal{O}_K$ is atomic.) Let $g(x) = (x - r_1)^2$ and $h(x) = (x - r_2) \cdots (x - r_q)$.

Let H, G of the same degree as g and h, respectively, be irreducible in $K[x]$ and non-associated in $K[x]$, such that for any product of copies of g and h, the fixed divisor is the same as that of any modified product in which some copies of g have been replaced by G and some copies of h by H. (That such G and H exist has been shown by some of the present authors together with R. Rissner [11, Lemma 3.3].)

Let

$$F(x) = \frac{G(x)H(x)}{p}.$$

Then F is irreducible in $\mathrm{Int}(\mathcal{O}_K)$, but not absolutely irreducible, because

$$F^2 = \frac{G(x)H(x)^2}{p^2} \cdot G(x).$$

□

Regarding non-absolutely irreducible elements of $\mathrm{Int}(\mathcal{O}_K)$, we can likewise generalize examples where the n-th power of an irreducible element has factorizations of length other than n (for instance, [24, Example 4.4]) from $\mathrm{Int}(\mathbb{Z})$ to $\mathrm{Int}(\mathcal{O}_K)$ by using [11, Lemma 3.3] as in the above proof.

5. Rings with non-absolutely irreducible elements and primes, but no other absolutely irreducible elements

Examples of atomic domains that have no absolutely irreducible elements, but contain both prime elements and non-absolutely irreducibles arise for instance from the D+M construction [1].

Example 5.1. Let $R = K_1 + xK_2[x]$, where $K_1 \subsetneq K_2$ are fields. Then R is atomic and its irreducible elements are of the form

(i) ax, where $a \in K_2$ or
(ii) $a(1 + xf(x))$, where $a \in K_1$, $f(x) \in K_2[x]$, and $1 + xf(x)$ is irreducible in $K_2[x]$, see [1].

Furthermore, every irreducible of the form $a(1 + xf(x))$ is prime [1]. The irreducible elements of the form ax are not absolutely irreducible, because

$$(ax)^2 = acx \cdot ac^{-1}x$$

with $c \in K_2 \setminus K_1$.

With a bit more effort, we can even construct Noetherian semilocal integral domains of this type.

Proposition 5.2. *Let D and V be one-dimensional Noetherian local integral domains that have a common quotient field L, and set $R = D \cap V$. Suppose further that the following hold:*

(i) *V is a discrete rank one valuation ring of L.*
(ii) *There exists a prime element π of V that is also a prime element of R.* [1]
(iii) *All the valuation rings that appear as localizations of the integral closure of D are independent of V.*
(iv) *D is not a unique factorization domain.*
(v) *There are at least two non-associated irreducible elements of D that are also irreducible in R.*

Then R is a one-dimensional Noetherian integral domain (so, in particular, an atomic domain) that has exactly two maximal ideals and precisely one prime element up to associativity, namely π. In addition, it has irreducible elements distinct from π and all of these are not absolutely irreducible.

[1] Of course, the prime element π of V is unique up to associativity *in* V. In the course of the proof we will see $\pi V \cap R = \pi R$, so all associates of π in V that are contained in R are also associated to π *in* R.

Proof. First, for the sake of completeness, we want to recall the full argument why D cannot be contained in V (which will be needed later in the proof). Assume to the contrary that $D \subseteq V$. Since V is integrally closed, the integral closure $\overline{D}$ of D is also contained in V.

Now there are two cases. In case $\pi V \cap \overline{D} = (0)$, all non-zero elements of $\overline{D} \setminus \{0\}$ are invertible in V and hence $L = (\overline{D} \setminus \{0\})^{-1}\overline{D} \subseteq V$, which is a contradiction. Otherwise, πV lies over a maximal ideal of $\overline{D}$ and hence $\overline{D}_{\pi V \cap \overline{D}} = V$. This is in contradiction to the assumption (iii). So, in total, we infer that D is not contained in V.

Denote by M the maximal ideal of D. We are now in the situation of [25, Theorem 3], which implies that $D = R_{M \cap R}$ and $V = R_{\pi V \cap R}$. Note that the prime ideals $M \cap R$ and $\pi V \cap R$ are not comparable by set-theoretic inclusion because, otherwise, D or V could not be a one-dimensional ring. So, we can apply [21, Theorem 105] and get that $M \cap R$ and $\pi V \cap R$ are exactly the maximal ideals of R. In particular, using the prime ideal correspondence under localization, R is a one-dimensional domain.

Because R is one-dimensional and π is a prime element of R by assumption, we must have $\pi R = \pi V \cap R$. In particular, $\pi \notin M$, and so π is a unit of D but a non-unit of V. Thus we are able to use [20, Corollary 1.20] and infer that R is Noetherian (and therefore atomic).

It is now left to show that the irreducible elements of R not associated to π are not absolutely irreducible. Since π is prime in R, all these irreducibles lie in $(M \cap R) \setminus (\pi V \cap R)$. Now, (v) says that there are at least two of them and hence, by [10, Lemma 2.1] they cannot be absolutely irreducible. □

We now give concrete examples of integral domains D and V as in 5.2. In particular, we thus find another atomic integral domain that has prime elements and irreducible elements that are not absolutely irreducible, but has no absolutely irreducible elements that are not prime.

Example 5.3. Let K be any field in which -1 is not a square, that is, a formally real field, for instance, $K = \mathbb{Q}$, and let X be an indeterminate over K. Define D and V as the localizations

$$D = K[X^2, X^3]_{(X^2, X^3)}, \; V = K[X]_{(X^2+1)},$$

and set $R = D \cap V$.

Note that $(x^2 + 1)$ is indeed a prime ideal of $K[X]$ since -1 is not a square in K. Both, D and V, are one-dimensional Noetherian local integral domains and their quotient field is just the function field $K(X)$. In order to see this for D, just note that $K[X^2, X^3]$ is a numerical semigroup algebra and hence one-dimensional and Noetherian.

Moreover, V clearly is a discrete rank one valuation ring with prime element $\pi = X^2 + 1$. The integral closure of D is the valuation ring $K[X]_{(X)}$ that is indeed independent of V.

Next we show that X^2+1 is also a prime element of R. For this, we can just argue that it generates the prime ideal $P = (X^2+1)_{(X^2+1)} \cap R$, where $(X^2+1)_{(X^2+1)}$ denotes the unique maximal ideal of V.

Let $g \in P$. As g is, a fortiori, an element of D, we can write it in the form $g = \frac{h}{s}$, where $h\ s \in K[X^2, X^3]$ and $s \notin (X^2, X^3)_{K[X^2,X^3]}$. On the other hand, g is also in $(X^2+1)_{(X^2+1)}$ and we can therefore write it as $g = \frac{a}{b}(X^2+1)$ with a, $b \in K[X]$ and $b \notin (X^2+1)_{K[X]}$.

It is our goal to show that $X^2 + 1$ divides g in R. Since $\frac{a}{b} \in V$ by its choice, it suffices to prove that $\frac{a}{b} \in D$. In order to do this, we clear denominators in the equation $\frac{h}{s} = \frac{a}{b}(X^2+1)$ and arrive at

$$bh = sa(X^2+1)$$

that we can view as an identity in $K[X]$. Since X^2+1 does not divide b in $K[X]$, it has to divide h and the unique cofactor is $\frac{sa}{b} \in K[X]$. Since $h \in K[X^2, X^3]$ has no linear term, it follows from $h = \frac{sa}{b}(X^2+1)$ that the polynomial $\frac{sa}{b}$ also has no linear term, that is, $\frac{sa}{b} \in K[X^2, X^3]$. By choice, $s \in K[X^2, X^3] \setminus (X^2, X^3)_{K[X^2,X^3]}$ and hence $\frac{a}{b} = \frac{1}{s} \cdot \frac{sa}{b} \in D$.

As a last step, we argue that the elements X^2 and X^3 are still irreducible in R. They are both elements of $(X^2, X^3)_{K[X^2,X^3]} \cap R$ and, therefore, non-units. In the following, for an irreducible polynomial p of $K[X]$, we denote by v_p the p-adic valuation on $K(X)$.

We carry out the argument for X^2; it is then analogous for X^3. So, decompose $X^2 = f \cdot g$ where f, $g \in R$. We want to show that either f or g is a unit of R. The elements of $K[X^2, X^3]_{(X^2,X^3)}$ (and therefore those of R) either have X-adic valuation 0 or ≥ 2. Since $\mathsf{v}_X(f) + \mathsf{v}_X(g) = \mathsf{v}_X(X^2) = 2$, we can assume without loss of generality that $\mathsf{v}_X(f) = 2$ and $\mathsf{v}_X(g) = 0$.

Furthermore, as elements of $K[X]_{(X^2+1)}$, the (X^2+1)-adic valuation of f and g is ≥ 0. Hence, the equality $0 = \mathsf{v}_{X^2+1}(X^2) = \mathsf{v}_{X^2+1}(f) + \mathsf{v}_{X^2+1}(g)$ implies that $\mathsf{v}_{X^2+1}(g) = 0$. To conclude, g is an element of R that is in none of the two maximal ideals of R and therefore a unit. This finishes the example.

6. Rings with prime elements, absolutely irreducible elements that are not prime, and non-absolutely irreducible elements

For an atomic domain that has prime elements, absolutely irreducible elements that are not prime, and non-absolutely irreducible elements, we consider certain Krull domains and subrings of the ring of integer-valued polynomials.

Lemma 6.1. *For each prime number p, the ring*

$$R(p) = \left\{ \frac{g}{p^n} \in \mathrm{Int}(\mathbb{Z}) \text{ with } g \in \mathbb{Z}[X] \text{ and } n \in \mathbb{N}_0 \right\}$$

has infinitely many prime elements. Furthermore, $R(p)$ has both absolutely irreducible elements that are not prime and non-absolutely irreducible elements.

Proof. First, each prime number $q \neq p$ is prime in $R(p)$: If q divides a product $(g_1/p^{n_1}) \cdots (g_r/p^{n_r})$ in $R(p)$, then q divides $g_1 \cdots g_r$ in $\mathbb{Z}[X]$. Because $\mathbb{Z}[X]$ is a UFD with q prime, without restriction $g_1 = qf_1$ with $f_1 \in \mathbb{Z}[X]$. For all $a \in \mathbb{Z}$, also $b := (qf_1(a))/p^{n_1} \in \mathbb{Z}$, and since $q \neq p$, it follows that $f_1(a)/p^{n_1} \in \mathbb{Z}$, showing $f_1/p^{n_1} \in R(p)$. We have therefore shown that q divides g_1/p^{n_1} in $R(p)$, so q is prime in $R(p)$.

Now, let $r_1, \ldots, r_p$ be a complete system of residues modulo p and set

$$f(x) = \frac{(x - r_1) \cdots (x - r_p)}{p}.$$

Then f is absolutely irreducible in $R(p)$. The polynomial f is not prime because f divides $(x - r_1) \cdots (x - r_p)$ but it does not divide any individual linear factor.

Furthermore, let $a_1, \ldots, a_p, b_{p+1}, \ldots, b_{p^2}$ be a complete system of residues modulo p^2 with $b_i \not\equiv 0 \pmod{p}$ for $p+1 \leq i \leq p^2$. Let $c_1, c_2 \in \mathbb{Z}$ such that $c_1 \equiv c_2 \equiv 0 \pmod{p^2}$ and $c_1 \neq c_2$. Set $g(x) = \prod_{k=p+1}^{p^2} (x - b_k)$ and

$$f(x) = \frac{g(x)(x - c_1)(x - c_2)^{e-1}}{p^e},$$

where $e = \mathsf{v}_p(p^2!) = p + 1$. Then f is irreducible in $R(p)$, but not absolutely irreducible, because

$$f^2 = \frac{g(x)(x - c_1)^2 (x - c_2)^{e-2}}{p^e} \cdot \frac{g(x)(x - c_2)^e}{p^e}$$

is a factorization of f^2 essentially different from $f \cdot f$. □

The non-absolutely irreducible elements of $R(p)$ whose n-th power has factorizations of different lengths can be constructed by adapting known examples in $\mathrm{Int}(\mathbb{Z})$ [24, Example 4.1 & Example 4.4].

Fact 6.2. *Let $\mathcal{O}_K$ be the ring of integers of a number field K. Then the following hold.*

(i) *$\mathcal{O}_K$ has infinitely many prime elements.*
(ii) *If $\mathcal{O}_K$ is not a unique factorization domain, then $\mathcal{O}_K$ has absolutely irreducible elements that are not prime, and non-absolutely irreducible elements, see* [10, Theorem 3.1] *or Corollary 6.6 below.*

Example 6.3. Let $R = \mathbb{Z}[\sqrt{d}]$ where $d < 0$ is a square-free composite integer not congruent to 1 modulo 4. Firstly, the element $\sqrt{d} = \sqrt{ec}$ is irreducible in R, but not absolutely irreducible, because $(\sqrt{d})^2 = e \cdot c$ is a non-trivial factorization of $(\sqrt{d})^2$ (not necessarily into irreducibles).

Secondly, let $r = a + b\sqrt{d} \in R$ and $N(r) = a^2 - db^2$, the norm of r. Then 2 is irreducible in R because $N(2) = 4$ and $N(r) \neq 2$ for all $r \in R$ (because $d \leq -6$). The element 2 is not prime in R because

(i) if $d \equiv 2 \pmod 4$, then $2 \,|\, (\sqrt{d})^2$, but $2 \not| \sqrt{d}$, and
(ii) if $d \equiv 3 \pmod 4$, then $2 \,|\, (1 + \sqrt{d})(1 - \sqrt{d})$, but 2 does not divide the individual factors.

More generally, by [22, Theorem 25], the prime decomposition of $2R$ is

(i) $2R = (2, \sqrt{d})^2$ if $d \equiv 2 \pmod 4$, and
(ii) $2R = (2, 1 + \sqrt{d})^2$ if $d \equiv 3 \pmod 4$.

It follows by [10, Theorem 3.1] that 2 is absolutely irreducible in R.

Lastly, it follows by [22, Theorem 25] that every odd prime p such that $p \not| d$ and $d^{\frac{p-1}{2}} \equiv -1 \pmod p$, is prime in R.

In the next subsection, we discuss absolute irreducibility in Krull monoids.

6.1. **Absolutely Irreducible Elements in Krull monoids.** In this section, let H be a Krull monoid with class group G. Let $G_0 \subseteq G$ denote the set of classes containing prime divisors, and let $G_1 \subseteq G_0$ denote the set of classes containing *exactly* one prime divisor.

Absolutely irreducible elements in Krull monoids have been characterized in various ways. The following extends [15, Proposition 7.1.4] (where $G = G_0$ is assumed) and parts of [18, Proposition 4.7] (where $H = \mathcal{B}(G_0)$ with G torsion-free). A similar characterization is given in [4, Lemma 5].

Proposition 6.4. *Let $\emptyset \neq S \subseteq \mathfrak{X}(H)$ be a finite set. The following statements are equivalent.*

(i) *There exists an absolute irreducible $a \in H$ with $\operatorname{supp}(aH) = S$.*
(ii) *The set S is minimal in $\{ \operatorname{supp}(bH) : b \in H \setminus H^\times \}$*
(iii) *The set S is minimal in $\{ \operatorname{supp}(bH) : b \in H$ is irreducible$\}$.*
(iv) *The family $([\mathfrak{p}])_{\mathfrak{p} \in S}$ in G is $\mathbb{Z}_{\geq 0}$-linearly dependent, and every proper subfamily is $\mathbb{Z}_{\geq 0}$-linearly independent.*
(v) *The family $([\mathfrak{p}])_{\mathfrak{p} \in S}$ in G is $\mathbb{Z}_{\geq 0}$-linearly dependent, and every proper subfamily is $\mathbb{Z}$-linearly independent.*

In case the equivalent conditions hold, the absolutely irreducible element with support S is uniquely determined up to associativity.

Proof. (i) ⇒ (ii) If $\operatorname{supp}(bH) \subsetneq S$, then $b \mid a^n$ for some $n \geq 1$ and b is not associated to a.

(ii) $\Rightarrow$ (iii) Trivial.

(iii) $\Rightarrow$ (ii) Suppose there exists $b \in H \setminus H^\times$ with $\operatorname{supp}(bH) \subsetneq S$. Let $u \in H$ be an irreducible element dividing b. Then $\operatorname{supp}(uH) \subsetneq S$.

(ii) $\Leftrightarrow$ (iv) Statement (iv) is just a explicit way of stating (ii).

(iv) $\Rightarrow$ (v) Fix a non-zero vector $(\alpha_\mathfrak{p})_{\mathfrak{p}\in S} \in \mathbb{Z}_{\geq 0}^S$ such that $\sum_{\mathfrak{p}\in S} \alpha_\mathfrak{p}[\mathfrak{p}] = 0$. Suppose, for the sake of contradiction, that there exists $S' \subsetneq S$ and a non-zero vector $(\beta_\mathfrak{p})_{\mathfrak{p}\in S'} \in \mathbb{Z}^{S'}$ such that $\sum_{\mathfrak{p}\in S'} \beta_\mathfrak{p}[\mathfrak{p}] = 0$. By the minimality of S, there must exist $\mathfrak{p} \in S'$ with $\beta_\mathfrak{p} < 0$. Let $\mathfrak{q} \in S'$ be such that $\beta_\mathfrak{q} < 0$ and so that $\alpha_\mathfrak{q}/\left|\beta_\mathfrak{q}\right|$ is minimal among all $\alpha_\mathfrak{p}/\left|\beta_\mathfrak{p}\right|$ with $\beta_\mathfrak{p} < 0$. Then

$$\left|\beta_\mathfrak{q}\right| \alpha_\mathfrak{p} + \alpha_\mathfrak{q}\beta_\mathfrak{p} \geq 0$$

for all $\mathfrak{p} \in S$, and equality holds for $\mathfrak{p} = \mathfrak{q}$. Since

$$\left|\beta_\mathfrak{q}\right| \sum_{\mathfrak{p}\in S} \alpha_\mathfrak{p}[\mathfrak{p}] + \alpha_\mathfrak{q} \sum_{\mathfrak{p}\in S'} \beta_\mathfrak{p}[\mathfrak{p}] = 0,$$

this contradicts the $\mathbb{Z}_{\geq 0}$-linear independence of $([\mathfrak{p}])_{\mathfrak{p}\in S\setminus\{\mathfrak{q}\}}$.

(v) $\Rightarrow$ (i) Consider the group homomorphism $\sigma\colon \mathbb{Z}^S \to G$ given by $\sigma((\alpha_\mathfrak{p})_{\mathfrak{p}\in S}) = \sum_{\mathfrak{p}\in S} \alpha_\mathfrak{p}[\mathfrak{p}]$. Our assumptions ensure that there exists some $(\alpha_\mathfrak{p})_{\mathfrak{p}\in S} \in \mathbb{Z}_{>0}^S \cap \ker(\sigma)$. Moreover, the image of σ contains a torsion-free subgroup of rank $|S| - 1$. Thus $\ker(\sigma)$ is free of rank one. Since we know that $\ker(\sigma)$ contains a vector with all positive coordinates, we can also choose a generator (as an abelian group) $(\beta_\mathfrak{p})_{\mathfrak{p}\in S}$ with all positive coordinates. The zero-sum sequences with support S correspond to elements of $\ker(\sigma)$ with positive entries, and they are all positive multiples of $(\beta_\mathfrak{p})_{\mathfrak{p}\in S}$.

Hence there is, up to associativity, a unique irreducible with support contained in S. This irreducible is then necessarily absolutely irreducible. ☐

While the absolute irreducibility of elements does not lift along the transfer homomorphism $H \to \mathcal{B}(G_0)$ in general (Remark 2.6), additional knowledge of the set G_1 nevertheless allows us to characterize when *every* irreducible is absolutely irreducible using $\mathcal{B}(G_0)$.

Theorem 2. *Let H be a Krull monoid with class group G, let G_0 be the set of classes containing prime divisors, and let G_1 be the set of classes containing precisely one prime divisor. The following are equivalent.*

(i) *Every irreducible of H is absolutely irreducible.*
(ii) *Every irreducible of $\mathcal{B}(G_0)$ is absolutely irreducible, and for every irreducible $U \in \mathcal{B}(G_0)$ and every $g \in G_0 \setminus G_1$ it holds that $\mathsf{v}_g(U) \leq 1$.*

Proof. Let $\theta\colon H \to \mathcal{B}(G_0)$ denote the block homomorphism.

(i) $\Rightarrow$ (ii) Since θ is a transfer homomorphism, the monoid $\mathcal{B}(G_0)$ also has the property that every irreducible is absolutely irreducible (by Lemma 2.5).

For the second property, suppose that there exists an irreducible $U \in \mathcal{B}(G_0)$ that is of the form $U = g^2T$ with $T \in \mathcal{F}(G_0)$, and that there exist $\mathfrak{p} \neq \mathfrak{q} \in \mathfrak{X}(H)$ such that $[\mathfrak{p}] = [\mathfrak{q}] = g$. We may assume $T = [\mathfrak{r}_1] \cdots [\mathfrak{r}_k]$ with $\mathfrak{r}_i \in \mathfrak{X}(H)$. Let $\mathfrak{a} = \mathfrak{r}_1 \cdots_v \mathfrak{r}_k$. Then $\mathfrak{p}^2 \cdot_v \mathfrak{a}$ and $\mathfrak{p} \cdot_v \mathfrak{q} \cdot_v \mathfrak{a}$ are principal ideals, say $aH = \mathfrak{p}^2 \cdot_v \mathfrak{a}$ and $bH = \mathfrak{p} \cdot_v \mathfrak{q} \cdot_v \mathfrak{a}$ with $a, b \in H$. Since θ is a transfer homomorphism, the irreducibility of U implies that of a and b. However $a \mid b^2$ and so b is not absolutely irreducible, contradicting our assumption.

(ii) ⇒ (i) Let $a \in H$ be irreducible. Then aH has a unique factorization

$$aH = \mathfrak{p}_1^{e_1} \cdots_v \mathfrak{p}_k^{e_k} \cdot_v \mathfrak{q}_1 \cdots_v \mathfrak{q}_l,$$

with pairwise distinct prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_k, \mathfrak{q}_1, \ldots, \mathfrak{q}_l$ and $e_i \geq 2$ for all $1 \leq i \leq k$. Suppose that $b \in H$ is an irreducible with $b \mid a^n$ for some $n \geq 1$. Then $\theta(b) \mid \theta(a)^n$. Since $\theta(a)$ is absolutely irreducible, we get $\theta(a) = \theta(b)$. By assumption $[\mathfrak{p}_i] \in G_1$ for all $1 \leq i \leq k$, so that $\theta(b)$ fully determines the multiplicity of each $\mathfrak{p}_i$ in bH. Now

$$bH = \mathfrak{p}_1^{e_1} \cdots_v \mathfrak{p}_k^{e_k} \cdot_v \mathfrak{q}_{i_1} \cdots_v \mathfrak{q}_{i_m},$$

for some $1 \leq i_1 \leq \cdots \leq i_m \leq l$. However, a simple length comparison shows $\{i_1, \ldots, i_m\} = \{1, \ldots, l\}$ and so $aH = bH$. □

Given the previous theorem, and the fact that we are often interested in cases when $G_0 = G$ and $G_1 = \emptyset$, it is useful to determine when every irreducible of $\mathcal{B}(G)$ is absolutely irreducible. The equivalence of the first two statements of the following lemma are well-known.

Lemma 6.5. *Let G be an abelian group. Then the following are equivalent for the monoid of zero-sum sequences $\mathcal{B}(G)$.*

(i) $|G| \leq 2$.
(ii) *$\mathcal{B}(G)$ is factorial.*
(iii) *Every irreducible of $\mathcal{B}(G)$ is absolutely irreducible.*

Proof. (i) ⇒ (ii): We recall the (well-known) argument. If G is trivial, then the sequence 0 (that is, the sequence of length 1 consisting of the single element $0 \in G$) is the only irreducible of $\mathcal{B}(G)$, and $\mathcal{B}(G) \cong \mathbb{N}_0$ is factorial. If $G = \{0, g\} \cong \mathbb{Z}/2\mathbb{Z}$, then 0 and g^2 are the only irreducibles of $\mathcal{B}(G)$, and they are both prime, so that $\mathcal{B}(G) \cong \mathbb{N}_0^2$.

(ii) ⇒ (iii): Trivial.

(iii) ⇒ (i): We prove the contrapositive. Assume $|G| \geq 3$. Then one of the following three cases must occur.

- *G contains an element g of finite order $n \geq 3$.* Consider the irreducibles $S = g^n$, $S' = (-g)^n$ and $T = g(-g) \in \mathcal{B}(G)$. Then $SS' = T^n$, and so T is not absolutely irreducible (note $g \neq -g$).
- *G contains two independent elements g, h both of order* 2. Consider the irreducibles $S = g^2$, $S' = h^2$, $S'' = (g + h)^2$, and $T = gh(g + h) \in \mathcal{B}(G)$. Then $SS'S'' = T^2$.

- *G contains an element g of infinite order.* Consider $S = (3g)(-g)^3$, $S' = (3g)^2(-2g)^3$ and $T = (-g)(-2g)(3g) \in \mathcal{B}(G)$. Then S, S', and T are irreducible and $SS' = T^3$. □

We note in passing that the statements of the previous lemma are further equivalent to $\mathcal{B}(G)$ being half-factorial.

The following (straightforwardly) generalizes the theorem of Chapman and Krause [10, Corollary 3.2], who proved the statement for $H = \mathcal{O}_K^\bullet$ with $\mathcal{O}_K$ a ring of algebraic integers in a number field.

Corollary 6.6. *Let H be a Krull monoid such that every class in the class group G contains a prime divisor (that is, $G = G_0$). Then every irreducible element of H is absolutely irreducible if and only if H is factorial.*

Proof. If H is factorial, then every irreducible is prime and hence absolutely irreducible.

For the converse, suppose that every irreducible of H is absolutely irreducible. By Theorem 2, every irreducible of $\mathcal{B}(G)$ must be absolutely irreducible. Thus $|G| \leq 2$ by Lemma 6.5. To show that G is trivial it therefore suffices to show $G \ncong \mathbb{Z}/2\mathbb{Z}$.

Suppose that $G = \{0, g\} \cong \mathbb{Z}/2\mathbb{Z}$. By (b) of [15, Theorem 2.5.4.1] there must exist at least two distinct non-zero divisorial prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of non-trivial class $[\mathfrak{p}] = [\mathfrak{q}] = g$. Therefore $\mathfrak{p}^2$, $\mathfrak{q}^2$ and $\mathfrak{p} \cdot_v \mathfrak{q}$ are all principal and generated by irreducibles u, v, w, say, $\mathfrak{p}^2 = uH$, $\mathfrak{q}^2 = vH$ and $\mathfrak{p}\mathfrak{q} = wH$. Now $w^2 \sim uv$ shows that w is not absolutely irreducible. □

We will see below, in Proposition 8.1, that some assumption on G_0 is necessary for the previous proposition. We also recover a theorem of Angermüller; see [5, Theorem 1(e)] for a generalization to monadically Krull monoids.

Theorem 3 ([4, Theorem 1(c)])**.** *A Krull monoid H is factorial if and only if every absolutely irreducible element is prime.*

Proof. If H is factorial, then every irreducible is prime.

For the converse, suppose that every absolutely irreducible element is prime. Suppose that H is not factorial. Then there exist non-prime irreducibles. Let $u \in H$ be such an irreducible with $\operatorname{supp}(uH)$ minimal among all irreducibles. Then u is absolutely irreducible by Proposition 6.4, and therefore prime, a contradiction. □

Remark 6.7. Absolutely irreducible elements in Krull monoids have been studied in different settings and under different names. For instance, in [18] they play a very central role in the setting of $H = \mathcal{B}(G_0)$ with G torsion-free, and are called *elementary atoms.* If H is a normal affine monoid, then the absolutely irreducible elements correspond precisely to the extremal rays of the polyhedral convex cone, whereas the irreducible elements form the Hilbert basis of the monoid. We refer to [13], in particular to §4 and therein to Remarks 13 and 16 for a discussion of the terminologies.

7. Rings with irreducible elements that are all absolutely irreducible, but none of them prime

Proposition 7.1. *There exists a Dedekind domain that is not half-factorial and such that all of its irreducible elements are absolutely irreducible but none of them prime.*

Proof. Let $n \geq 2$ be an integer, let $G = \mathbb{Z}^n$, and let $\{e_1, e_2, \ldots, e_n\}$ the standard $\mathbb{Z}$-basis for G. Define $f = \sum_{i=1}^{n} e_i$ and set

$$G_0 = \{\pm e_i,\ \pm f\}\,.$$

Consider the monoid of zero-sum sequences $\mathcal{B}(G_0)$ over G_0. The irreducible elements of $\mathcal{B}(G_0)$ are

$$\{e_i(-e_i),\ f(-f),\ e_1e_2\cdots e_n(-f),\ (-e_1)(-e_2)\cdots(-e_n)f\}\,.$$

Because the supports of these sequences are pairwise incomparable, all of them are absolutely irreducible.

Let

$$U = e_1e_2\cdots e_n(-f) \quad \text{and} \quad V = (-e_1)(-e_2)\cdots(-e_n)f.$$

Then in $\mathcal{B}(G_0)$ we have the non-unique factorization

$$UV = (e_1(-e_1))\cdot(e_2(-e_2))\cdots(e_n(-e_n))\cdot f(-f).$$

It follows from [17, Theorem 8] that there exists a Dedekind domain D with class group $G = \mathbb{Z}^n$ and G_0 precisely the set of classes containing prime ideals. There exists a transfer homomorphism $\varphi\colon D \setminus \{0\} \to \mathcal{B}(G_0)$ (see Theorem 1). In particular, the domain D is not half-factorial and an element $a \in D$ is irreducible if and only if the corresponding zero-sum sequence $\varphi(a) \in \mathcal{B}(G_0)$ is irreducible.

Since $0 \notin G_0$, the trivial ideal class of D contains no prime ideals. Hence D contains no prime element. Finally, every irreducible element of $\mathcal{B}(G_0)$ is square-free, and so Theorem 2 implies that every irreducible element of D is absolutely irreducible. □

8. Rings with all irreducible elements absolutely irreducible but not all prime

In contrast to Proposition 7.1, the following result gives an analogous example of a Dedekind domain, but this time it contains a prime element.

Proposition 8.1. *There exists a Dedekind domain D that is not half-factorial, contains a prime element, and such that all of its irreducibles elements are absolutely irreducible.*

Proof. Repeat the proof of Proposition 7.1 with the set

$$G_0 = \left\{0, \pm e_i,\ \sum_{i=1}^{n} e_i,\ \sum_{i=1}^{n} -e_i\right\}.$$

Note that $0 \in G_0$ and hence there exists a non-zero principal prime ideal in D and therefore a prime element. □

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