A NOTE ON CANCELLATION IN TOTALLY DEFINITE QUATERNION ALGEBRAS

DANIEL SMERTNIG

Abstract. In [8, 4] it is shown that there exist only finitely many isomorphism classes of Eichler orders of square-free level in totally definite quaternion algebras over number fields having locally free cancellation, and they are all classified. There is an error in this classification which is corrected in the present note.

1. Introduction

In [8], Vignéras showed that only finitely many isomorphism classes of Eichler orders of square-free level in a totally definite quaternion algebra over a number field can have the cancellation property for locally free modules. She provided a criterion (called Vignéras' criterion in the sequel) characterizing this property, determined that locally free cancellation is only possible over number fields of degree at most 33, and, in the case of quadratic and cubic abelian fields, classified the Eichler orders of square-free level having locally free cancellation. Recently, in [4], Hallouin and Maire finished the classification using Vignéras' criterion, by first improving the degree bound to 8, and then analyzing the remaining cases.

A first reduction step used in the proof of the criterion is that an order $\mathcal{O}$ in a quaternion algebra over a number field has locally free cancellation if and only if every stably free (left) $\mathcal{O}$-module is free. In this short note we point out that this is not true, as it is necessary but not sufficient for every stably free $\mathcal{O}$-module to be free. Vignéras' criterion can be used to determine when every stably free left $\mathcal{O}$-ideal is principal, and hence when every stably free $\mathcal{O}$-module is free. We discuss in detail an example of a maximal order $\mathcal{O}$ that has the property that every stably free $\mathcal{O}$-module is free, but does not have locally free cancellation (Section 3).

The bounds obtained in [4] for degree and discriminant of a number field K, over which orders $\mathcal{O}$ in totally definite quaternion algebras with locally free cancellation exist, consequently are in fact bounds for the property that every stably free $\mathcal{O}$-module is free. They are such that all totally real number fields in question have been enumerated by Voight ([10]). Using the computer algebra system Magma ([1]), that implements algorithms of Kirschmer and Voight for computations in quaternion algebras ([6]), it is possible to enumerate all Eichler orders of square-free level which have locally free cancellation and those which have the weaker property that every stably free module is free. Thereby we correct the classification of Eichler orders with locally free cancellation, and supplement it by one of those Eichler orders in which every stably free module is free. Where the resulting tables differ from the original classification, we give explanations for the deviations when this is possible.

Since this is a short note based on [8] and [4], we do not recall all the concepts and notation, instead referring the reader to the two papers in question. All unexplained notation is as in [4]. Background about quaternion algebras can be found in [9, 7]. Throughout the paper, all modules are assumed to be finitely generated.

2010 *Mathematics Subject Classification.* 11R52, 16H10, 20G30.
Key words and phrases. totally definite quaternion algebras, Eichler orders, cancellation property.
This work was supported by the Austrian Science Fund (FWF): W1230.

2. The original classification

In this section we recall the notion of locally free cancellation for an Eichler order $\mathcal{O}$ of square-free level, and point out how it differs from the property that every stably free left $\mathcal{O}$-module is free.

Let K be a totally real number field, A its ring of integers, $\mathcal{H}$ a totally definite quaternion algebra with center K, and $\mathcal{O}$ an Eichler (A-)order of square-free level of $\mathcal{H}$. As in [9, I.4], an ideal I of $\mathcal{H}$ is a full A-sublattice of $\mathcal{H}$, and I is a left $\mathcal{O}$-ideal if $\mathcal{O}_l(I) = \{ x \in \mathcal{H} \mid xI \subset I \} = \mathcal{O}$. Since the Eichler orders of square-free level are precisely the hereditary orders, every left $\mathcal{O}$-ideal is projective and hence locally principal (this follows from [5, Theorem 1 and 2]). (However, if $\mathcal{O}$ is not maximal, then, due to the extra condition $\mathcal{O}_l(I) = \mathcal{O}$, not every nonzero left ideal of $\mathcal{O}$ is a left $\mathcal{O}$-ideal.)

If $M \neq \mathbf{0}$ is a finitely generated, locally free $\mathcal{O}$-module, then $M \cong \mathcal{O}^n \oplus I$ with $n \in \mathbb{N}_0$ and a locally principal left ideal I of $\mathcal{O}$ ([3, §I]). We may assume $I \neq \mathbf{0}$. Then I is a full A-sublattice of $\mathcal{H}$, and since I is locally principal, $\mathcal{O}_l(I)_\mathfrak{p} = \mathcal{O}_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{spec}(A)$, and hence $\mathcal{O}_l(I) = \mathcal{O}$. Thus I is in fact a left $\mathcal{O}$-ideal.

$\mathcal{O}$ has *locally free cancellation* if and only if $M \oplus \mathcal{O}^m \cong N \oplus \mathcal{O}^m$ with finitely generated, locally free left $\mathcal{O}$-modules M, N and $m \in \mathbb{N}_0$ implies $M \cong N$. By the discussion in the previous paragraph, this is equivalent to the same statement with M and N replaced by left $\mathcal{O}$-ideals.

A left $\mathcal{O}$-module M is *stably free* if there exist $m, n \in \mathbb{N}_0$ such that $M \oplus \mathcal{O}^m \cong \mathcal{O}^n$. If M is stably free, then it is locally free because locally the Krull-Schmidt-Azumaya Theorem holds ([2, Theorem 30.18]). Again, every stably free left $\mathcal{O}$-module is free if and only if every stably free left $\mathcal{O}$-ideal is principal. Clearly this holds if $\mathcal{O}$ has locally free cancellation.

Two left $\mathcal{O}$-modules M and N are *stably isomorphic* if there exists $m \in \mathbb{N}_0$ such that $M \oplus \mathcal{O}^m \cong N \oplus \mathcal{O}^m$. The stable isomorphism classes of locally free left $\mathcal{O}$-ideals form a finite abelian group, the locally free class group of $\mathcal{O}$. Let $\mathcal{C}^+(A)$ denote the narrow class group of A, which is defined to be the factor group of all non-zero fractional ideals of A modulo the principal fractional ideals generated by totally positive elements of K. Since $\mathcal{O}$ is an Eichler order, the locally free class group of $\mathcal{O}$ is, via the reduced norm map, isomorphic to $\mathcal{C}^+(A)$ (using [3, §X]). Denoting by $\mathrm{LF}_1(\mathcal{O})$ the set of isomorphism classes of left $\mathcal{O}$-ideals, we therefore have a map of finite sets

$$\mu_\mathcal{O} : \begin{cases} \mathrm{LF}_1(\mathcal{O}) & \to \mathcal{C}^+(A) \\ [I] & \mapsto [\mathrm{Nrd}(I)]. \end{cases}$$

This map is always surjective, and $\mathcal{O}$ has locally free cancellation if and only if $\mu_\mathcal{O}$ is injective (hence bijective). $\mathcal{O}$ has the property that every stably free left $\mathcal{O}$-ideal is principal if and only $\mu_\mathcal{O}^{-1}(\{\mathbf{0}\}) = \{ [\mathcal{O}] \}$. (See [3, Section 3].) Because $\mu_\mathcal{O}$ is merely a map of finite sets, these two properties are in general not equivalent. Instead we have the following.

Lemma 1. *The following statements are equivalent.*

(a) *$\mathcal{O}$ has locally free cancellation.*
(b) *For every Eichler order $\mathcal{O}'$ in $\mathcal{H}$ of the same level as $\mathcal{O}$, every stably free left $\mathcal{O}'$-ideal is principal.*

Proof. (a) $\Rightarrow$ (b): If $\mathcal{O}$ has locally free cancellation, then so do all the other Eichler orders with the same level (by [8, Proposition 4e and f]). Locally free cancellation implies that every stably free left $\mathcal{O}'$-ideal is principal.

(b) $\Rightarrow$ (a): It suffices to show that two stably isomorphic left $\mathcal{O}$-ideals are isomorphic. Let I, J be left $\mathcal{O}$-ideals, and assume that they are stably isomorphic. Since $\mathcal{O}_r(J^{-1}) = \mathcal{O} = \mathcal{O}_l(I)$, the product $J^{-1}I$ is proper, and

$$\mu_\mathcal{O}([J^{-1}I]) = [\mathrm{Nrd}(J^{-1}I)] = [\mathrm{Nrd}(J)^{-1}\,\mathrm{Nrd}(I)] = -[\mathrm{Nrd}(J)] + [\mathrm{Nrd}(I)] = \mathbf{0} \in \mathcal{C}^+(A).$$

(We write $\mathcal{C}^+(A)$ additively.) Hence $J^{-1}I$ is a stably free left $\mathcal{O}_r(J)$-ideal. Since $\mathcal{O}_r(J)$ is an Eichler order of the same level as $\mathcal{O}$, it follows that $J^{-1}I$ is principal. Thus $J \cong I$. □

Note that in the proof above $\mathcal{O}_r(J)$ may not be isomorphic to $\mathcal{O}$, and therefore the proof does not work if one merely requires that stably free left $\mathcal{O}$-ideals are principal. It can in fact happen that this property holds for some but not all orders (see Section 3). Vignéras' original criterion (cf. [8, Théorème 3] or [4, Theorem 10]) can be used to determine whether a given Eichler order $\mathcal{O}$ of square-free level has the property that every stably free left $\mathcal{O}$-ideal is principal, but to determine whether or not locally free cancellation holds it must be applied to a representative of each of the isomorphism classes of Eichler orders of a given level.

The classification of Eichler orders of square-free level that have locally free cancellation in [8] and [4] is carried out under the assumption that it suffices to check the criterion for just one order of a given discriminant and level. This leaves a gap in the classification. Furthermore, it raises the question of finding the full list of Eichler orders $\mathcal{O}$ of square-free level that do not possess locally free cancellation, but that do have the weaker property that every stably free left $\mathcal{O}$-module is free.

3. An explicit example

In this section we give an example of a maximal order $\mathcal{O}$ that has the property that every stably free left $\mathcal{O}$-module is free, but that does not have locally free cancellation.

Let $K = \mathbb{Q}(\sqrt{6})$, and let $A = \mathbb{Z}[\sqrt{6}]$ be its ring of algebraic integers. Then A is a PID (i.e., $h = 1$), and its narrow class number is $h^+ = 2$. Let $\mathcal{H} = {}_K\langle 1, i, j, k\rangle$ be the quaternion algebra with $i^2 = -1$, $j^2 = -1$, and $ij = -ji = k$.

Let

$$\begin{aligned}
\mathcal{O}_1 &= {}_A\Big\langle 1,\ j,\ \frac{1}{2} + \frac{1}{2}i + \frac{\sqrt{6}}{2}j,\ \frac{\sqrt{6}}{2} + \frac{1}{2}j - \frac{1}{2}k\Big\rangle,\\
\mathcal{O}_2 &= {}_A\Big\langle 1,\ \frac{1}{2}(2-\sqrt{6})(1+i),\ \frac{1}{2}(2-\sqrt{6})(1+j),\ \frac{1}{2}(1+i+j+k)\Big\rangle,\\
\mathcal{O}_3 &= {}_A\Big\langle 1,\ i,\ \frac{1}{2}(\sqrt{6}-3)(1+i) + \frac{1}{2}(2-\sqrt{6})j,\ \frac{1}{2}(\sqrt{6}-3) + \frac{1}{4}(2-\sqrt{6})(j+k)\Big\rangle.
\end{aligned}$$

By inspection, the given generators are linearly independent over K, so $KO_m = \mathcal{H}$ for all $m \in [1,3]$. Moreover, for all $m \in [1,3]$, $1 \in \mathcal{O}_m$ and a straightforward computation shows that $\mathcal{O}_m$ is multipliciatively closed (it suffices to check that the product of any two basis elements lies in $\mathcal{O}_m$ again), showing that $\mathcal{O}_m$ is a ring, hence an order in $\mathcal{H}$. If $u_1, \ldots, u_4$ denotes the given basis elements for $\mathcal{O}_m$, then the square of the discriminiant of $\mathcal{O}_m$ is the ideal generated by $\det((\mathrm{Trd}(u_m u_n))_{m,n\in[1,4]})$ ([9, Lemma I.4.7(3)]). Again, a straightforward calculation shows that this determinant is in A^*, hence $\mathcal{O}_m$ is maximal. [1]

By applying Vignéras' criterion, we will show that every stably free left $\mathcal{O}_1$-ideal is principal, but that this is not so for stably free left $\mathcal{O}_2$-ideals and stably free left $\mathcal{O}_3$-ideals.

The torsion subgroups $\mathcal{W}_m$ of $\mathcal{O}_m^*$ must be isomorphic to one of C_{2n}, H_{4n} or E_{24}, E_{48}, E_{120} (cf. [4, §1.3]). Since $\sqrt{2} \notin K$ and $\sqrt{5} \notin K$, [4, Lemma 4] implies that no subgroup of $\mathcal{H}^*$ is isomorphic to E_{48} or E_{120}. If $n > 6$, then $\phi(2n) > 4$, so that $[\mathbb{Q}(\zeta_{2n})^+ : \mathbb{Q}] > 2$. Moreover, none of $\mathbb{Q}(\zeta_8)^+ = \mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\zeta_{10})^+ = \mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\zeta_{12})^+ = \mathbb{Q}(\sqrt{3})$ embed into K either. By [4, Lemma 4], therefore no subgroup of $\mathcal{H}^*$ is isomorphic to H_{4n} for $n > 3$.

[1] In fact, $\mathcal{H}$ has three conjugacy classes of maximal orders, and three isomorphism classes of left $\mathcal{O}_m$-ideals. In principle it seems feasible to check this by hand using the Eichler mass formula (e.g., [9, Corollaire V.2.5] or [6, Proposition 5.1]), but in practice it is easier to use Magma for the calculation. The given orders are pairwise non-isomorphic due to their different torsion subgroups (as we will see), and hence they represent all conjugacy classes of maximal orders. We will however not make use of this fact.

We claim that the torsion subgroups are as follows.

$$\begin{aligned} H_8 &\cong \mathcal{W}_1 = \Big\{ \pm 1,\, \pm i,\, \pm j,\, \pm k \Big\}, \\ E_{24} &\cong \mathcal{W}_2 = \Big\{ \pm 1,\, \pm i,\, \pm j,\, \pm k,\, \frac{\pm 1 \pm i \pm j \pm k}{2} \Big\}, \\ H_{12} &\cong \mathcal{W}_3 = \Big\{ \pm 1,\, \pm i,\, \frac{\pm 2 \pm \sqrt{6}(j+k)}{4},\, \frac{\pm 2i \pm \sqrt{6}(j-k)}{4} \Big\}. \end{aligned}$$

Again, a straightforward computation shows that the stated elements lie in the torsion subgroups of $\mathcal{O}_1^*, \mathcal{O}_2^*, \mathcal{O}_3^*$. It remains to check that no other elements are in $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$. For $\mathcal{W}_2$ and $\mathcal{W}_3$ this follows because certainly H_{12} and E_{24} are not contained in a cyclic group, no subgroup of E_{24} is isomorphic to H_{12}, and we have already seen that H_{4n} for $n > 3$ and E_{48}, E_{120} are not isomorphic to any subgroup of $\mathcal{H}^*$. Assume that $|\mathcal{W}_1| > 8$. Then necessarily $\mathcal{W}_1 \cong E_{24}$, and therefore $\mathcal{W}_1$ is conjugate to $\mathcal{W}_2$ under conjugation by an element of $\mathcal{H}^*$. The only elements of order 2 of $\mathcal{W}_2$ are $\{\pm i, \pm j, \pm k\}$ and this set must therefore be taken to itself under the conjugation. From this we conclude $\mathcal{W}_1 = \mathcal{W}_2$ and, in particular, $\frac{1}{2}(1 + i + j + k) \in \mathcal{W}_1 \subset \mathcal{O}_1$. From the given basis for $\mathcal{O}_1$ it is immediate that this is false, and hence we must have $\mathcal{W}_1 \cong H_8$.

We apply Vignéras' criterion, in the form that a maximal order $\mathcal{O}_i$ has the property that every stably free left $\mathcal{O}_i$-ideal is principal if and only if

$$\zeta_K(-1)\Phi(d_1, d_2) = \frac{2^n[(A^*)^+ : (A^*)^2]}{|\mathcal{W}_i|[\mathcal{O}_i^* : \mathcal{W}_i A^*]}$$

(cf. [4, Theorem 10]). Since $\zeta_K(-1) = \frac{1}{2}$, $\Phi(d_1, d_2) = 1$, $n = [K : \mathbb{Q}] = 2$ and $[(A^*)^+ : (A^*)^2] = \frac{h^+}{h} = 2$, the criterion reduces to

$$16 = |\mathcal{W}_i|[\mathcal{O}_i^* : \mathcal{W}_i A^*].$$

Because $|\mathcal{W}_2| = 24$ and $|\mathcal{W}_3| = 12$, neither of which divides 16, obviously $\mathcal{O}_2$ and $\mathcal{O}_3$ do not have the property that every stably free left $\mathcal{O}_2$-ideal [left $\mathcal{O}_3$-ideal] is principal. For $\mathcal{O}_1$ however $|\mathcal{W}_1| = 8$ and we claim $[\mathcal{O}_1^* : \mathcal{W}_1 A^*] = 2$, which implies that every stably free left $\mathcal{O}_1$-ideal is principal: Set $\varepsilon = (\frac{\sqrt{6}}{2} - 1)(1 + i) \in \mathcal{O}_1$. Then $\mathrm{Nrd}(\varepsilon) = 5 - 2\sqrt{6} \in A^*$, hence $\varepsilon \in \mathcal{O}_1^*$, and clearly $\varepsilon \in (1 + i)K^*$. A short computation shows $5 - 2\sqrt{6} \notin (K^*)^2$, and hence [4, Proposition 6.3] implies $[\mathcal{O}_1^* \mathcal{W}_1 : A^*] = 2$, as claimed.

4. The revised classification

Let K be a totally real number field of degree n, $\mathcal{H}$ a totally definite quaternion algebra with center K, and let $\mathcal{O}$ be an Eichler order of square-free level in $\mathcal{H}$, such that every stably free left $\mathcal{O}$-ideal is principal. Vignéras showed ([8, p.267, eqn. (21)])

$$\mathrm{rd}_K = |\mathrm{dis}(K)|^{\frac{1}{n}} \le 4^{1-\frac{2}{3n}} \pi^{\frac{4}{3}},$$

and Hallouin and Maire have shown that necessarily $n \le 8$ (from the proof of [4, Proposition 11]), and gave even stronger bounds on rd_K if $n \le 6$ ([4, Proposition 12]). In any case, $\mathrm{rd}_K < 17$, and all totally real number fields up to this degree and root discriminant have been computed by Voight (cf. [10] and [11] for the extended tables).

With the help of the computer algebra system Magma [1] it is therefore easily possible to iterate over all candidate orders, to compute the set of isomorphism classes of left $\mathcal{O}$-ideals, $\mathrm{LF}_1(\mathcal{O})$, the narrow class group, $\mathcal{C}^+(A)$, and the map $\mu_{\mathcal{O}}\colon \mathrm{LF}_1(\mathcal{O}) \to \mathcal{C}^+(A)$ explicitly, and to thereby determine the orders $\mathcal{O}$ having locally free cancellation, as well as those that do not have locally free cancellation but satisfy the weaker property that every stably free left $\mathcal{O}$-module is principal.

Based on such a computation we describe in the last two subsections first corrections to the classification of Eichler orders of square-free levels having locally free cancellation, and then the invariants of those Eichler orders of square-free level that do not possess locally free cancellation, but still possess the weaker property that every stably free left module is free.

4.1. **Locally free cancellation.** The following invariants correspond to Eichler orders of square–free level over quadratic fields that have locally free cancellation and are missing from the table in [8, Théorème 8]. (1 stands for the principal A-ideal generated by 1).

d_K	(d_1, d_2)
5	$(1, \mathfrak{p}_2\mathfrak{p}_5)$, $(1, \mathfrak{p}_2\mathfrak{p}_{11}^{(i)})$
8	$(1, \mathfrak{p}_2\mathfrak{p}_7^{(i)})$
12	$(1, \mathfrak{p}_2\mathfrak{p}_3)$, $(1, \mathfrak{p}_{23}^{(i)})$, $(\mathfrak{p}_2\mathfrak{p}_3, \mathfrak{p}_{11}^{(i)})$
21	$(\mathfrak{p}_2\mathfrak{p}_5^{(i)}, 1)$, $(1, \mathfrak{p}_3)$
28	$(\mathfrak{p}_2\mathfrak{p}_3^{(i)}, 1)$

Moreover, the table in [8] contains an entry for the invariant $(\mathfrak{p}_2\mathfrak{p}_3, 1)$ over the quadratic field with discriminant $d_K = 60$. This order however does not have the cancellation property: This follows from $\zeta_K(-1) = 2$, $\Phi(d_1, d_2) = 2$, $[\mathcal{O}^* : A^*] = 2$, $[A^{*+} : A^{*2}] = 2$, $|\mathcal{W}| = 2$ and $[\mathcal{O}^* : \mathcal{W}A^*] = 2$. (Or: $h(\mathcal{O}) = 8$ while for the narrow class number we have $h^+(A) = 4$.)

For fields of higher degree, Hallouin and Maire tabulate the Eichler orders with locally free cancellation in [4, Theorem 13]. It turns out that the only entry in their table not actually possessing locally free cancellation is the algebra over a field of degree 6. It appears in the table because it has one isomorphism class of maximal orders for which every stably free left module is free, and they happen to be checking the criterion for an order from this class. However, there are three isomorphism classes of maximal orders, and the other two do not have this property.

For the field K with defining polynomial $f = x^4 - x^3 - 4x^2 + 4x + 1$ ($d_K = 1125$) some entries are omitted in the table: The invariants $(1, \mathfrak{q}_{29})$, $(1, \mathfrak{r}_{29})$ and $(1, \mathfrak{s}_{29})$ seem to be missing from the final table by a simple oversight as it is proven in the paper that orders with these invariants possess locally free cancellation.

Moreover, the orders with invariants $(1, \mathfrak{p}_5)$ and $(1, \mathfrak{p}_9)$ are missing from the table. The invariant $(1, \mathfrak{p}_5)$ is discarded in [4, p.207, second paragraph] as not having locally free cancellation by arguing that ζ_5 cannot be contained in an Eichler order $\mathcal{O}$ with this invariant, while this would have to be the case for locally free cancellation to hold. To argue $\zeta_5 \notin \mathcal{O}$, implicitly [4, Proposition 2] is used. But this proposition applies to the ring of integers of $K(\zeta_5)$ while $A[\zeta_5]$ is a non-maximal order in $K(\zeta_5)$, so that the argument breaks down. (A more refined version of the proposition that takes into account non-maximal orders is [9, Corollaire III.5.12].) The case $(1, \mathfrak{p}_9)$ is completely analogous.

4.2. **Stably free implies free.** The following table describes the invariants of Eichler orders $\mathcal{O}$ of square-free level that do not have locally free cancellation, but that do have the property that every stably free left $\mathcal{O}$-module is free. The first four columns describe a field K: f denotes a polynomial such that $K \cong \mathbb{Q}[x]/(f)$, d_K is the discriminant of K, $h(A)$ and $h^+(A)$ are the class number and the narrow class number of the ring of integers A of K. The next column, (d_1, d_2), is an invariant (discriminant and level) of an Eichler order, with the property that at least one of the Eichler orders with this invariant has the property that every stably free left module is free. Following that, $h(\mathcal{O})$ is the number of left ideal classes of an Eichler order of this discriminant and level, and $t(\mathcal{O})$ is the number of conjugacy classes of such Eichler orders. The final column, labeled "#", states how many of these $t(\mathcal{O})$ conjugacy classes have the property that every stably free module is free (with one exception, this is always just one). Generators for the orders in question are not given, as they tend to be lengthy and can easily be recomputed from the given information using Magma.

f	d_K	$h(A)$	$h^+(A)$	(d_1, d_2)	$h(\mathcal{O})$	$t(\mathcal{O})$	#
$x^6 - 3x^5 - 3x^4 + 10x^3 + 3x^2 - 6x + 1$	1397493	1	2	$(1, 1)$	6	6	1
$x^6 - x^5 - 6x^4 + 6x^3 + 8x^2 - 8x + 1$	453789	1	2	$(1, 1)$	3	3	1
$x^4 - 7x^2 - 6x + 1$	10512	1	4	$(1, 1)$	8	8	2
$x^4 - 6x^2 - 3x + 3$	9909	1	2	$(1, 1)$	5	5	1
$x^4 - 2x^3 - 6x^2 + 7x + 11$	5125	1	2	$(1, 1)$	4	4	1
$x^4 - 2x^3 - 3x^2 + 4x + 1$	4752	1	2	$(1, 1)$	4	4	1
$x^4 - 6x^2 - 4x + 2$	4352	1	2	$(1, 1)$	4	4	1
				$(1, \mathfrak{p}_2)$	6	4	1
$x^4 - 4x^2 + 1$	2304	1	2	$(1, 1)$	3	3	1
$x^4 - 5x^2 + 5$	2000	1	2	$(1, 1)$	3	3	1
				$(1, \mathfrak{p}_5)$	4	3	1
$x^2 - 69$	69	1	2	$(1, 1)$	5	5	1
$x^2 - 15$	60	2	4	$(1, 1)$	8	8	1
$x^2 - 33$	33	1	2	$(1, 1)$	3	3	1
$x^2 - 7$	28	1	2	$(1, 1)$	3	3	1
				$(1, \mathfrak{p}_2)$	4	3	1
$x^2 - 6$	24	1	2	$(1, 1)$	3	3	1
				$(1, \mathfrak{p}_3)$	4	3	1

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Institut für Mathematik und wissenschaftliches Rechnen, Karl-Franzens-Universität Graz, Heinrichstrasse 36, 8010 Graz, Austria

E-mail address: `daniel.smertnig@uni-graz.at`