FACTORIZATION IN THE SELF-IDEALIZATION OF A PID

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Abstract. Let D be a principal ideal domain and $R(D) = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D\}$ be its self-idealization. It is known that $R(D)$ is a commutative noetherian ring with identity, and hence $R(D)$ is atomic (i.e., every nonzero nonunit can be written as a finite product of irreducible elements). In this paper, we completely characterize the irreducible elements of $R(D)$. We then use this result to show how to factorize each nonzero nonunit of $R(D)$ into irreducible elements. We show that every irreducible element of $R(D)$ is a primary element, and we determine the system of sets of lengths of $R(D)$.

1. Introduction

Let R be a commutative noetherian ring. Then R is atomic, which means that every nonzero nonunit element of R can be written as a finite product of atoms (irreducible elements) of R. The study of non-unique factorizations has found a lot of attention. Indeed this area has developed into a flourishing branch of Commutative Algebra (see some surveys and books [2, 6, 8, 5]). However, the focus so far was almost entirely on commutative integral domains, and only first steps were done to study factorization properties in rings with zero-divisors (see [3, 7]). In the present note we study factorizations in a subring of a matrix ring over a principal ideal domain, which will turn out to be a commutative noetherian ring with zero-divisors.

To begin with, we fix our notation and terminology. Let R be a commutative ring with identity and $U(R)$ be the set of units of R. Two elements $a, b \in R$ are said to be *associates* if $aR = bR$. Clearly, if $a = ub$ for some $u \in U(R)$, then a and b are associates. An $a \in R$ is said to be *irreducible* if $a = bc$ implies that either b or c is associated with a. We say that R is *atomic* if every nonzero nonunit of R is a finite product of irreducible elements. It is clear that noetherian rings are atomic (cf. [3, Theorem 3.2]) and that $0 \in R$ is irreducible if and only if R is an integral domain. A ring R is a half-factorial ring (HFR) (resp., bounded factorization ring (BFR)) if R is atomic and two factorizations of a nonzero nonunit into irreducible elements have the same length (resp., for each nonzero nonunit $x \in R$, there is an integer $N(x) \geq 1$ so that for any factorization $x = x_1 \cdots x_n$, where each x_i is a nonunit, we have $n \leq N(x)$). R is a FFR (finite factorization ring) if R is atomic and each nonzero nonunit has only finitely many factorizations into irreducibles, up to order and associates. A nonzero nonunit $x \in R$ is said to be prime (resp., primary) if xR is a prime (resp., primary) ideal. Hence a prime element is primary but not vice

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versa (for example, if $\mathbb{Z}$ is the ring of integers, then $4 \in \mathbb{Z}$ is primary but not prime). We say that R is a *unique factorization ring* (UFR) if every nonzero principal ideal of R can be written as a product of principal prime ideals (cf. [3, Theorem 4.9]). Clearly, a prime element is irreducible, and so a UFR is atomic.

For $x \in R$ a nonzero nonunit, its *set of lengths* is defined as

$$\mathsf{L}(x) = \{\, k \in \mathbb{N} \mid \text{there exist irreducibles } u_1, \ldots, u_k \in R \text{ with } x = u_1 \cdot \ldots \cdot u_k \,\}.$$

Clearly, x is irreducible if and only if $\mathsf{L}(x) = \{\,1\,\}$. If $x \in U(R)$, we set $\mathsf{L}(x) = \{\,0\,\}$. The *system of sets of lengths* is defined as $\mathcal{L}(R) = \big\{\, \mathsf{L}(x) \mid x \in R \setminus \{\,0\,\} \,\big\}$. Sets of lengths and invariants derived from them are some of the classical invariants considered in non-unique factorization theory (see [8, Ch. 1.4]). The reader is referred to [8] for undefined definitions and notations.

Let M be an R-module. The *idealization* $R(+)M$ of M is a ring, which is defined as an abelian group $R \oplus M$, whose ring multiplication is given by $(a, b) \cdot (x, y) = (ax, ay + bx)$ for all $a, x \in R$ and $b, y \in M$. It is known that $R(+)M$ is a noetherian ring if and only if R is noetherian and M is finitely generated [4, Theorem 4.8]. Let D be an integral domain, $\mathrm{Mat}_{2\times 2}(D)$ be the ring of 2×2 matrices over D, and $R(D) = \{ \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D\}$. It is easy to show that $R(D)$ is a commutative ring with identity under the usual matrix addition and multiplication; in particular, $R(D)$ is a subring of $\mathrm{Mat}_{2\times 2}(D)$. Clearly, the map $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ embeds D into $R(D)$, and the map $\varphi : D(+)D \to R(D)$, given by $\varphi(a, b) = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$, is a ring isomorphism. In view of this isomorphism, $R(D)$ is called the self-idealization of D (cf. [13]). There is also an isomorphism $D[X]/\langle X^2 \rangle \to R(D)$ mapping $a + bX + \langle X^2 \rangle$ to $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$. Some factorization properties of $R(+)M$ have been studied in [3, Theorem 5.2]. For more on basic properties of $R(+)M$ (and hence of $R(D)$), see [4] or [11, Section 25].

Let D be a principal ideal domain (PID). Then $R(D)$ is noetherian, and thus $R(D)$ is atomic. In Section 2, we first characterize the irreducible elements of $R(D)$, and we then use this result to show how to factorize each nonzero nonunit of $R(D)$ into irreducible elements via the factorization of D. We show that $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is the unique prime element (up to associates) of $R(D)$. We prove that every nonzero nonunit of $R(D)$ can be written as a product of primary elements. Finally, in Section 3, we completely describe the system of sets of lengths $\mathcal{L}(R(D))$.

We write $\mathbb{N} = \{\,1, 2, 3, \ldots\,\}$ for the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{\,0\,\}$ for the set of non-negative integers.

2. Factorization in $R(D)$ when D is a PID

Let D be an integral domain, and

$$R(D) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, a, b \in D \right\}$$

be the self-idealization of D. Clearly, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is the identity of $R(D)$.

If $\alpha = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \in R(D)$, then $\mathrm{nr}(\alpha) = a$ is the *norm*, and this is a ring homomorphism $R(D) \to D$. Observe that α is a zero-divisor if and only if $a = 0$. We write $R(D)^\bullet$ for the monoid of non-zero-divisors of $R(D)$.

We begin this paper by characterizing the units of $R(D)$, which is very useful in the proof of Theorem 5.

Lemma 1. (cf. [11, Theorem 25.1(6)]) *If D is an integral domain, then $\alpha = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \in R(D)$ is a unit of $R(D)$ if and only if a is a unit of D.*

Proof. If α is a unit, then

$$\alpha \cdot \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for some} \quad \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \in R(D).$$

Thus $ax = 1$, and so $a \in U(D)$. Conversely, assume that a is a unit, and let $u = a^{-1}$. Then

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} u & -bu^2 \\ 0 & u \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u & -bu^2 \\ 0 & u \end{pmatrix} \in R(D).$$

Thus α is a unit. □

For an arbitrary commutative ring R, there can be two elements $a, b \in R$ such that a and b are associates but $a \neq ub$ for all $u \in U(R)$ (see, for example, [3, Example 2.3]). This cannot happen in the self-idealization of an integral domain.

Lemma 2. *Let D be an integral domain and $\alpha, \beta \in R(D)$ and let $a, b, x, y \in D$ such that $\alpha = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$ and $\beta = \left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$. The following statements are equivalent.*

(a) *α and β are associates.*
(b) *There exists $\theta \in U(R(D))$ such that $\beta = \theta\alpha$.*
(c) *There exists $u \in U(D)$ such that $x = au$ and $y \equiv bu \mod a$.*

Proof. (a) ⇒ (b): If α and β are associates, then there are some $\gamma, \delta \in R(D)$ so that $\alpha = \beta\gamma$ and $\beta = \alpha\delta$. Hence if

$$\gamma = \begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} x_1 & y_1 \\ 0 & x_1 \end{pmatrix},$$

then $a = xa_1$ and $x = ax_1$, and so $a_1, x_1 \in U(D)$. Thus $\gamma, \delta \in U(R(D))$ by Lemma 1.

(b) ⇒ (c): Let $\theta = \left(\begin{smallmatrix} u & v \\ 0 & u \end{smallmatrix}\right)$. By Lemma 1, $u \in U(D)$. From $\beta = \theta\alpha$ it follows that $x = au$ and $y = av + bu \equiv bu \mod a$.

(c) ⇒ (b) and (a): Let $v \in D$ be such that $y = bu + av$. Define $\theta = \left(\begin{smallmatrix} u & v \\ 0 & u \end{smallmatrix}\right)$. Then $\theta \in U(R(D))$ by Lemma 1 and $\beta = \theta\alpha$. Thus, α and β are associates. □

We write $\alpha \simeq \beta$ if $\alpha, \beta \in R(D)$ are associates.

Lemma 3. *Let D be a PID and let $\alpha = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \in R(D)^\bullet$. If $a = cd$ with coprime $c, d \in D$, then there exist $\gamma, \delta \in R(D)$ with $\alpha = \gamma\delta$ and $\mathrm{nr}(\gamma) = c$, $\mathrm{nr}(\delta) = d$. This representation is unique in the sense that, if $\gamma', \delta' \in R(D)$ with $\alpha = \gamma'\delta'$ and $\mathrm{nr}(\gamma') \simeq c$, $\mathrm{nr}(\delta') \simeq d$, then $\gamma \simeq \gamma'$ and $\delta \simeq \delta'$.*

Proof. Existence: Since $1 \in \mathrm{GCD}(c, d)$ and D is a PID, there exist $e, f \in D$ such that $b = cf + ed$. Then $\gamma = \left(\begin{smallmatrix} c & e \\ 0 & c \end{smallmatrix}\right)$ and $\delta = \left(\begin{smallmatrix} d & f \\ 0 & d \end{smallmatrix}\right)$ are as claimed.

Uniqueness: Let

$$\gamma' = \begin{pmatrix} c' & e' \\ 0 & c' \end{pmatrix} \quad \text{and} \quad \delta' = \begin{pmatrix} d' & f' \\ 0 & d' \end{pmatrix}$$

with $c', e', d', f' \in D$ and suppose that $\alpha = \gamma'\delta'$. Let $u, v \in U(D)$ such that $c' = cu$ and $d' = dv$. Since $c'd' = cd$, necessarily $v = u^{-1}$. Since $cf + ed = c'f' + e'd' =$

$c(f'u) + d(e'v)$, we have $c(f'u) \equiv cf \mod d$ and $f'u \equiv f \mod d$, i.e., $f' \equiv fv \mod d$. Therefore $\delta' \simeq \delta$ and similarly $\gamma' \simeq \gamma$. □

Corollary 4. *Let D be a PID and let $\alpha \in R(D)^\bullet \setminus U(R(D))$. Then there exist $\beta_1, \ldots, \beta_n \in R(D)^\bullet$ of pairwise distinct prime power norm, such that $\alpha = \beta_1 \cdot \ldots \cdot \beta_n$. This representation is unique up to order and associates.*

Proof. Let $\mathrm{nr}(\alpha) = p_1^{e_1} \cdot \ldots \cdot p_n^{e_n}$ with $n \geq 0$, $p_1, \ldots, p_n \in D$ pairwise distinct prime elements and $e_1, \ldots, e_n \geq 1$. By induction on n and the previous lemma, there exist $\beta_1, \ldots, \beta_n \in R(D)^\bullet$ such that $\alpha = \beta_1 \cdot \ldots \cdot \beta_n$ and $\mathrm{nr}(\beta_i) = p_i^{e_i}$ for all $i \in [1, n]$.

Suppose $\alpha = \beta_1' \cdot \ldots \beta_m'$ is another such factorization. Since D is a UFD, then $m = n$ and there exists a permutation $\pi \in \mathfrak{S}_n$ such that $\mathrm{nr}(\beta'_{\pi(i)}) \simeq \mathrm{nr}(\beta_i)$ for all $i \in [1, n]$. The uniquenes statement of the previous lemma implies $\beta_i' \simeq \beta_i$ for all $i \in [1, n]$. □

As a consequence, to study factorizations of $\alpha \in R(D)^\bullet$, it is sufficient to study factorizations of $\alpha \in R(D)^\bullet$ with prime power norm.

We next give the first main result of this paper, which completely characterizes the irreducible elements of $R(D)$ when D is a PID.

Theorem 5. *Let D be a PID and $\alpha = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \in R(D)$. Then α is irreducible if and only if either (i) $a = 0$ and $b \in U(D)$, (ii) $a = p$ or (iii) $a = up^n$ and $1 \in \mathrm{GCD}(a, b)$ for some prime $p \in D$, $u \in U(D)$, and integer $n \geq 2$.*

Proof. Necessity. Assume that $a = 0$, and let $\beta = \left(\begin{smallmatrix} b & 0 \\ 0 & b \end{smallmatrix}\right)$ and $\gamma = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then $\alpha = \beta \cdot \gamma$ and $\alpha R(D) \neq \beta R(D)$ because $b \neq 0$. Hence $\alpha R(D) = \gamma R(D)$, and so $\gamma = \alpha \cdot \delta$ for some $\delta = \left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) \in R(D)$. Thus $bx = 1$.

Next, assume that $a \neq 0$. If a is not of the form up^n, then Lemma 3 implies that $\alpha = \beta \cdot \gamma$ with $\mathrm{nr}(\beta)$ and $\mathrm{nr}(\gamma)$ nonzero nonunits. Hence α is not irreducible, a contradiction. Thus $a = up^n$ for some prime $p \in D$, $u \in U(D)$, and integer $n \geq 1$. If $n = 1$, then up is also a prime element of D and we have case (ii).

Finally, assume that $n \geq 2$ and $p^k \in \mathrm{GCD}(a, b)$ for some integer $k \geq 1$. Let $b = b_1 p^k$, where $b_1 \in D$. Then $\alpha = \theta \cdot \xi$, where

$$\theta = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \quad \text{and} \quad \xi = \begin{pmatrix} up^{n-1} & b_1 p^{k-1} \\ 0 & up^{n-1} \end{pmatrix},$$

but $\theta, \xi \notin \alpha R(D)$, a contradiction. This completes the proof.

Sufficiency. Let $\alpha = \beta \cdot \gamma$, where

$$\beta = \begin{pmatrix} x_1 & y_1 \\ 0 & x_1 \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} x_2 & y_2 \\ 0 & x_2 \end{pmatrix}.$$

We will show that β or γ is a unit, and thus α is irreducible.

Case 1. $a = 0$ and $b \in U(D)$. Note that $x_1 = 0$ or $x_2 = 0$; so for convenience, let $x_2 = 0$. Then $x_1 y_2 = b$, and since $b \in U(D)$, we have $x_1 \in U(D)$. Thus β is a unit of $R(D)$ by Lemma 1.

Case 2. $a = p$ for a prime $p \in D$. Then $\alpha = \beta \cdot \gamma$ implies that either x_1 or x_2 is a unit in D. Hence β or γ is a unit in $R(D)$ by Lemma 1.

Case 3. $a = up^n$ for a prime $p \in D$, $u \in U(D)$, $n \geq 2$ and $1 \in \mathrm{GCD}(a, b)$. Since p is a prime and $\alpha = \beta \cdot \gamma$, we have

$$\beta = \begin{pmatrix} vp^k & x \\ 0 & vp^k \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} wp^{n-k} & y \\ 0 & wp^{n-k} \end{pmatrix}$$

for some $0 \leq k, n-k \leq n$, $x, y \in D$, and $v, w \in U(D)$ with $vw = u$. Hence $b = p^k vy + p^{n-k} wx$, and thus $k = 0$ or $n - k = 0$ because a and b are coprime. Therefore β or γ is a unit in $R(D)$ by Lemma 1. □

Corollary 6. *Let D be a PID and $\alpha = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \in R(D)$ be a nonzero nonunit such that $c \in \mathrm{GCD}(a, b)$, $a = ca_1$, and $b = cb_1$ for some $c, a_1, b_1 \in D$. Let $c = up_1^{e_1} \cdots p_n^{e_n}$ and $a_1 = q_1^{k_1} \cdots q_m^{k_m}$ (when $a \neq 0$) be prime factorizations of c and a_1, respectively, where $u \in U(D)$. The following is a factorization of α into irreducible elements.*

(1) *If $a = 0$, then*

$$\alpha = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \prod_{i=1}^{n} \begin{pmatrix} p_i & 0 \\ 0 & p_i \end{pmatrix}^{e_i}.$$

(2) *If $a \neq 0$, then*

$$\alpha = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \prod_{i=1}^{n} \begin{pmatrix} p_i & 0 \\ 0 & p_i \end{pmatrix}^{e_i} \prod_{j=1}^{m} \begin{pmatrix} q_j^{k_j} & c_j \\ 0 & q_j^{k_j} \end{pmatrix}$$

for some $c_j \in D$ with $1 \in \mathrm{GCD}(c_j, q_j)$.

Proof. (1) Clear.

(2) We first note that

$$\alpha = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix}$$

and

$$\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} p_1 & 0 \\ 0 & p_1 \end{pmatrix}^{e_1} \cdots \begin{pmatrix} p_n & 0 \\ 0 & p_n \end{pmatrix}^{e_n}.$$

Next, assume that $a_1 = b_2 d_2$ for some $b_2, d_2 \in D$ with $1 \in \mathrm{GCD}(b_2, d_2)$. Then there are some $x, y \in D$ such that $b_2(xb_1) + d_2(yb_1) = b_1$ because D is a PID, and hence

$$\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} = \begin{pmatrix} b_2 & yb_1 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} d_2 & xb_1 \\ 0 & d_2 \end{pmatrix}.$$

Note that $1 \in \mathrm{GCD}(a_1, b_1)$; hence $1 \in \mathrm{GCD}(b_2, yb_1)$ and $1 \in \mathrm{GCD}(d_2, xb_1)$. So by repeating this process, we have

$$\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} = \prod_{j=1}^{m} \begin{pmatrix} q_j^{k_j} & c_j \\ 0 & q_j^{k_j} \end{pmatrix}$$

for some $c_j \in D$ with $1 \in \mathrm{GCD}(c_j, q_j)$. □

Corollary 7. *If D is a PID, then $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is the unique prime element (up to associates) of $R(D)$.*

Proof. Clearly, prime elements are irreducible, and hence by Theorem 5, we have three cases to consider. Let $\alpha = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \in R(D)$ be irreducible.

Case 1. $a = 0$ and $b \in U(D)$. Note that if we set $I = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, then $\alpha = I \cdot \left(\begin{smallmatrix} b & 0 \\ 0 & b \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} b & 0 \\ 0 & b \end{smallmatrix}\right) \in U(R(D))$ by Lemma 1; so α and I are associates. Let $\beta = \left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right), \gamma = \left(\begin{smallmatrix} c & d \\ 0 & c \end{smallmatrix}\right) \in R(D)$. Then $\beta\gamma \in IR(D)$ if and only if $xc = 0$; so if $x = 0$ (for convenience), then $\beta \in IR(D)$. Thus I is a prime.

Cases 2 and 3. $a \neq 0$. Note that

$$\begin{aligned} \begin{pmatrix} a & b-1 \\ 0 & a \end{pmatrix}^2 &= \begin{pmatrix} a^2 & 2a(b-1) \\ 0 & a^2 \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} a & b-2 \\ 0 & a \end{pmatrix} \in \alpha R(D), \end{aligned}$$

but $\left(\begin{smallmatrix} a & b-1 \\ 0 & a \end{smallmatrix}\right) \notin \alpha R(D)$ because $a \notin U(D)$. Thus α is not a prime. □

For zero-divisors and elements with prime power norm, the following lemma further refines Corollary 6, by giving all possible factorizations, up to order and associates. The general case can be obtained in combination with Corollary 4.

Lemma 8. *Let D be a PID, and let $\alpha = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \in R(D) \setminus \{ 0 \}$ with $a, b \in D$.*

(1) *Suppose $a = 0$ and $b = q_1 \cdot \ldots \cdot q_n$, with (possibly associated) prime powers $q_1, \ldots, q_n \in D$. Then, for every choice of $a_1, \ldots, a_n \in D$,*

$$\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \prod_{i=1}^{n} \begin{pmatrix} q_i & a_i \\ 0 & q_i \end{pmatrix},$$

and this is a factorization into irreducibles if and only if for all $i \in [1, n]$ either q_i is prime or $1 \in \mathrm{GCD}(q_i, a_i)$.

(2) *Suppose $a = p^n$ with $p \in D$ a prime element and $n \in \mathbb{N}$. For all $l \in [1, n]$ let $m_l \in \mathbb{N}_0$ and for all $j \in [1, m_l]$ let $a_{l,j} \in D$. Then*

$$\alpha = \prod_{l=1}^{n} \prod_{j=1}^{m_l} \begin{pmatrix} p^l & a_{l,j} \\ 0 & p^l \end{pmatrix}$$

if and only if $n = \sum_{l=1}^{n} m_l l$ and $b = \sum_{l=1}^{n} p^{n-l} (\sum_{j=1}^{m_l} a_{l,j})$. This is a product of irreducibles if and only if $1 \in \mathrm{GCD}(a_{l,j}, p)$ for all $l \in [2, n]$ and $j \in [1, m_l]$.

Up to order and associativity of the factors, all the factorizations of α are of this form.

Proof. This is checked by a straightforward calculation. The statement about the irreducibles follows from the characterization of the irreducible elements in Theorem 5. That every representation of α as a product of irreducibles is, up to order and associates, one of the stated ones also follows from this characterization. □

Corollary 9. *Let D be a PID.*

(1) *$R(D)$ is a BFR.*
(2) *$R(D)$ is a FFR if and only if D/pD is finite for all prime elements $p \in D$.*
(3) *If D is a field, then every nonzero nonunit of $R(D)$ is a prime, and hence $R(D)$ is a UFR with a unique nonzero (prime) ideal.*

Proof. (1) By Corollary 6, $R(D)$ is atomic, and if $\alpha = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \in R(D)$, then the lengths of factorizations of α into irreducible elements are less than or equal to that of the prime factorization of a or b in D, plus at most one. Thus $R(D)$ is a BFR.

(2) Suppose first that D/pD is finite for all prime elements $p \in D$. Then also D/p^nD is finite for all $n \geq 1$ and all prime elements $p \in D$. Hence, by the Chinese Remainder Theorem, D/cD is finite for all nonzero $c \in D$.

Let $c \in D^\bullet$. By Lemma 2(c) there exist, up to associativity, only finitely many elements $\gamma \in R(D)$ with $\mathrm{nr}(\gamma) \simeq c$. If $\alpha \in R(D)^\bullet$ and $\gamma|\alpha$, then $\mathrm{nr}(\gamma)|\,\mathrm{nr}(\alpha)$, and therefore there are, up to associativity, only finitely many irreducibles that can possibly divide α. Together with (1), this implies that every $\alpha \in R(D)^\bullet$ has only finitely many factorizations.

If $\alpha = \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \in R(D)$ is a zero-divisor, then every factorization has exactly one factor associated to $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and if γ is any other factor in the factorization then $\mathrm{nr}(\gamma) \mid b$ (cf. Lemma 8(1)). By the same argument as before, α has only finitely many factorizations.

For the converse, suppose that $p \in D$ is a prime element and $|D/pD| = \infty$. Since

$$\begin{pmatrix} p^2 & 0 \\ 0 & p^2 \end{pmatrix} = \begin{pmatrix} p & a \\ 0 & p \end{pmatrix} \begin{pmatrix} p & -a \\ 0 & p \end{pmatrix},$$

for all $a \in D$, $\begin{pmatrix} p^2 & 0 \\ 0 & p^2 \end{pmatrix}$ has infinitely many (non-associated) factorizations in $R(D)$.

(3) Let $\alpha = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \in R(D)$ be a nonzero nonunit. Since D is a field, by Lemma 1, $a = 0$ and $b \in U(D)$. Hence α is associated with $I := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and so α is a prime by the proof of Corollary 7. Thus $R(D)$ is a UFR and $IR(D)$ is a unique nonzero (prime) ideal of $R(D)$. ☐

If D is a PID but not a field, we will see in Corollary 15 that $R(D)$ is not a UFR, even when D is the ring of integers.

We next prove that every nonunit of $R(D)$ can be written as a (finite) product of primary elements.

Lemma 10. *Let R be a commutative ring. If $a \in R$ is such that $\sqrt{aR}$ is a maximal ideal, then aR is primary.*

Proof. Let $x, y \in R$ be such that $xy \in aR$ but $x \notin \sqrt{aR}$. Note that $\sqrt{aR} \subsetneq \sqrt{aR + xR}$; so $aR + xR = \sqrt{aR + xR} = R$ because $\sqrt{aR}$ is a maximal ideal. Hence $1 = as + xt$ for some $s, t \in R$. Thus $y = y(as + xt) = a(ys) + (xy)t \in aR$. ☐

Corollary 11. *If D is a PID, then every irreducible element of $R(D)$ is primary. In particular, each nonzero nonunit of $R(D)$ can be written as a finite product of primary elements.*

Proof. Let $\alpha = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \in R(D)$ be irreducible. By Theorem 5, there are three cases that we have to consider.

Case 1. $a = 0$ and $b \in U(D)$. By Corollary 7, α is a prime, and hence a primary element.

Cases 2 and 3. $a = up^n$ for some prime element $p \in D$, $u \in U(D)$, and $n \in \mathbb{N}$. By Lemma 10, it suffices to show that $\sqrt{\alpha R(D)}$ is a maximal ideal. Let $\beta = \left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) \in R(D) \setminus \sqrt{\alpha R(D)}$. Note that if $\delta = \left(\begin{smallmatrix} 0 & d \\ 0 & 0 \end{smallmatrix}\right) \in R(D)$, then $\delta^2 = 0$, and hence

$\delta \in \sqrt{\alpha R(D)}$. Hence

$$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \notin \sqrt{\alpha R(D)} \quad \text{and} \quad \begin{pmatrix} up^n & 0 \\ 0 & up^n \end{pmatrix} \in \sqrt{\alpha R(D)}.$$

But then $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right) \in \sqrt{\alpha R(D)}$. Note also that if $x \in pD$, then $x = px_1$ for some $x_1 \in D$, and so

$$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} x_1 & 0 \\ 0 & x_1 \end{pmatrix} \in \sqrt{\alpha R(D)},$$

a contradiction. So $x \notin pD$, and hence $xz_1 + pz_2 = 1$ for some $z_1, z_2 \in D$. Thus

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \beta \cdot \begin{pmatrix} z_1 & 0 \\ 0 & z_1 \end{pmatrix} + \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} z_2 & 0 \\ 0 & z_2 \end{pmatrix} + \begin{pmatrix} 0 & -yz_1 \\ 0 & 0 \end{pmatrix} \in \beta R(D) + \sqrt{\alpha R(D)}.$$

Therefore $\sqrt{\alpha R(D)}$ is maximal. □

Remark 12. In view of Corollary 11, Corollary 4 in fact corresponds to the (unique) primary decomposition of $\alpha R(D)$, as every prime ideal of $R(D)$, except for $0(+)D$, is maximal (cf. [4, Theorem 3.2]).

Associativity is a congruence relation on $(R(D)^\bullet, \cdot)$, and we denote by $R(D)^\bullet_{\mathrm{red}}$ the corresponding quotient monoid. Corollary 4 may also be viewed as a monoid isomorphism $R(D)^\bullet_{\mathrm{red}} \cong \coprod_p R(D_{(p)})^\bullet_{\mathrm{red}}$, where the coproduct is taken over all associativity classes of prime elements p of D, and $D_{(p)}$ is the localization at pD.

3. The sets of lengths in $R(D)$ when D is a PID

Let D be an integral domain and $R(D) = \{ \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D\}$. In this section, we characterize the sets of lengths in $R(D)$ when D is a PID.

Lemma 13. *Let D be a PID and $\alpha, \beta \in R(D)$.*

(1) *If $\alpha\beta \neq 0$, then $\mathsf{L}(\alpha) + \mathsf{L}(\beta) \subset \mathsf{L}(\alpha\beta)$.*
(2) *If $\mathrm{nr}(\alpha)$ and $\mathrm{nr}(\beta)$ are coprime, then $\mathsf{L}(\alpha) + \mathsf{L}(\beta) = \mathsf{L}(\alpha\beta)$.*

Proof. (1) Clear.

(2) Let $n \in \mathsf{L}(\alpha\beta)$. Then there exist irreducible $\gamma_1, \ldots, \gamma_n \in R(D)^\bullet$ such that $\alpha\beta = \gamma_1 \cdot \ldots \cdot \gamma_n$. Then also $\mathrm{nr}(\alpha)\,\mathrm{nr}(\beta) = \mathrm{nr}(\gamma_1) \cdot \ldots \cdot \mathrm{nr}(\gamma_n)$. Since $1 \in \mathrm{GCD}(\mathrm{nr}(\alpha), \mathrm{nr}(\beta))$, we may without loss of generality assume $\mathrm{nr}(\alpha) \simeq \mathrm{nr}(\gamma_1) \cdot \ldots \cdot \mathrm{nr}(\gamma_k)$ and $\mathrm{nr}(\beta) \simeq \mathrm{nr}(\gamma_{k+1}) \cdot \ldots \cdot \mathrm{nr}(\gamma_n)$ for some $k \in [0, n]$. By Lemma 3, therefore $\alpha \simeq \gamma_1 \cdot \ldots \cdot \gamma_k$ and $\beta \simeq \gamma_{k+1} \cdot \ldots \cdot \gamma_n$, and $n = k + (n-k) \in \mathsf{L}(\alpha) + \mathsf{L}(\beta)$. □

For a prime element $p \in D$ we denote by $\mathsf{v}_p \colon D \to \mathbb{N}_0 \cup \{\infty\}$ the corresponding valuation, i.e., $\mathsf{v}_p(0) = \infty$ and $\mathsf{v}_p(ap^k) = k$ if $k \in \mathbb{N}_0$ and $a \in D^\bullet$ is coprime to p.

Theorem 14. *Let D be a PID, $\alpha \in R(D)$ and suppose $\alpha = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$ with $a, b \in D$.*

(1) *If $a = 0$, and $b = p_1^{e_1} \cdot \ldots \cdot p_n^{e_n}$ with pairwise non-associated prime elements $p_1, \ldots, p_n \in D$ and $e_1, \ldots, e_n \in \mathbb{N}$, then $\mathsf{L}(\alpha) = [1+n, 1+e_1+\ldots+e_n]$.*
(2) *Let $p \in D$ be a prime element, $n \in \mathbb{N}$ and suppose $a = p^n$ and $\mathsf{v}_p(b) = k \in \mathbb{N}_0 \cup \{\infty\}$. Then $\mathsf{L}(\alpha) = \{1\}$ if and only if $k = 0$ or $n = 1$. If $k \geq n-1$, then*

$$[3, n-2] \cup \{n\} \subset \mathsf{L}(\alpha) \subset [2, n-2] \cup \{n\},$$

and if $k \in [1, n-2]$*, then*

$$[3, k+1] \subset \mathsf{L}(\alpha) \subset [2, k+1].$$

Moreover, if $k \geq 1$*, then* $2 \in \mathsf{L}(\alpha)$ *if and only if* n *is even or* $k < \frac{n}{2}$*.*

Proof. (1) This is clear from Lemma 8(1), as every factorization of b into prime powers gives a factorization of α (choose $a_i = 1$), and conversely.

(2) The cases $k = 0$ and $n = 1$ are clear from Theorem 5, so from now on we assume $k \geq 1$ and $n > 1$. Let $b = up^k$ with $u \in D$ and $1 \in \mathrm{GCD}(u, p)$. We repeatedly make use of Lemma 8(2), and the notation used there to describe a factorization, without explicitly mentioning this fact every time.

Claim A: $\mathsf{L}(\alpha) \subset [2, \min\{\, k+1,\, n\,\}]$.

Proof. Because α is not an atom, $1 \notin \mathsf{L}(\alpha)$. Any factorization of α is associated to one in Lemma 8(2); we fix a factorization of α with notation as in the lemma. The length of the factorization is then $t = \sum_{l=1}^{n} m_l$. Since $\sum_{l=1}^{n} m_l l = n$, clearly $t \leq n$. Moreover, necessarily $m_l = 0$ for all $l > n-(t-1)$. Since $b = \sum_{l=1}^{n} p^{n-l}(\sum_{j=1}^{m_l} a_{l,j})$, therefore $k = \mathsf{v}_p(b) \geq \mathsf{v}_p(p^{n-(n-t+1)}) = t-1$, i.e., $t \leq k+1$.

Claim B: $2 \in \mathsf{L}(\alpha)$ if and only if n is even or $k < \frac{n}{2}$.

Proof. Suppose $2 \in \mathsf{L}(\alpha)$ and n is odd. Then $n = l + (n-l)$ and $b = p^{n-l}a_{l,1} + p^l a_{n-l,1}$ with $1 \in \mathrm{GCD}(a_{l,1}, p)$ and $1 \in \mathrm{GCD}(a_{n-l,1}, p)$. Since n is odd, then $n - l \neq l$ and therefore $k = \mathsf{v}_p(b) = \min\{\, n-l,\, l\,\} < \frac{n}{2}$.

For the converse suppose first $1 \leq k < \frac{n}{2}$. Then $n = k + (n-k)$, $n - k > k$ and $b = p^{n-k} \cdot 1 + p^k(u - p^{n-2k})$ with $1 \in \mathrm{GCD}(u - p^{n-2k}, p)$. If n is even and $k \geq \frac{n}{2}$, then $n = \frac{n}{2} + \frac{n}{2}$ and $b = p^{\frac{n}{2}}(1 + (up^{k-\frac{n}{2}} - 1))$ with $1 \in \mathrm{GCD}(up^{k-\frac{n}{2}} - 1, p)$.

Claim C: If $n - 1 \in \mathsf{L}(\alpha)$, then $k = n-2$.

Proof. For a corresponding factorization we must have $m_1 = n-2$, $m_2 = 1$, and $m_l = 0$ for all $l > 2$. Then $b = p^{n-1}(a_{1,1} + \ldots + a_{1,n-2}) + p^{n-2}a_{2,1}$ with $1 \in \mathrm{GCD}(a_{2,1}, p)$, whence $k = \mathsf{v}_p(b) = n-2$.

Claim D: Let $n \geq 3$ and $k \geq 2$. If either $k = 2$ or $n \neq 4$, then $3 \in \mathsf{L}(\alpha)$.

Proof. Suppose first that n is odd and set $b' = b/p$. Then

$$\alpha = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \alpha' \quad \text{with} \quad \alpha' = \begin{pmatrix} p^{n-1} & b' \\ 0 & p^{n-1} \end{pmatrix}, \tag{1}$$

and, by Claim B, $2 \in \mathsf{L}(\alpha')$. Therefore $3 \in \mathsf{L}(\alpha)$.

If n is even, $n \geq 6$, and $k \geq 3$, then

$$\alpha = \begin{pmatrix} p^2 & u \\ 0 & p^2 \end{pmatrix} \begin{pmatrix} p^{n-2} & u(p^{k-2} - p^{n-4}) \\ 0 & p^{n-2} \end{pmatrix},$$

where the first factor is irreducible and the second has a factorization of length 2 by Claim B.

If $k = 2$, then

$$\alpha = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^2 \begin{pmatrix} p^{n-2} & u \\ 0 & p^{n-2} \end{pmatrix}$$

is a factorization of length 3.

Claim E: If $k \geq n-1$, then $n \in \mathsf{L}(\alpha)$.

Proof. We use Lemma 8(2). Set $m_1 = n$, $a_{1,1} = up^{k-(n-1)}$ and $a_{1,2} = \ldots = a_{l,n} = 0$. Then $p^{n-1}(up^{k-(n-1)} + 0 + \ldots + 0) = b$.

Claim F: If $k \in [1, n-2]$, then $[3, k+1] \subset \mathsf{L}(\alpha)$.

Proof. If $n \leq 3$ or $k = 1$, then the claim is trivially true, so we may assume $k \geq 2$. We proceed by induction on n. Suppose $n \geq 4$, and that the claim is true for $n-1$.

Let $b' = b/p$ and let α' be as in (1). We have $\mathsf{v}_p(b') = k - 1 \geq 1$.

If $k = 2$, then $1 = k - 1 < \frac{n-1}{2}$, and hence $2 \in \mathsf{L}(\alpha')$ (by Claim B). Therefore $\{3\} = [3, k+1] \subset \{1\} + \mathsf{L}(\alpha') \subset \mathsf{L}(\alpha)$.

If $k \geq 3$, then by induction hypothesis, $[3, k] \subset \mathsf{L}(\alpha')$, and thus $[4, k+1] = \{1\} + \mathsf{L}(\alpha') \subset \mathsf{L}(\alpha)$, and by Claim D, also $3 \in \mathsf{L}(\alpha)$.

Claim G: If $k \geq n-1$, then $[3, n-2] \subset \mathsf{L}(\alpha)$.

Proof. If $n \leq 4$, then the claim is trivially true. We again proceed by induction on n. Suppose $n \geq 5$ (then $k \geq 4$), and that the claim is true for $n-1$.

Let $b' = b/p$ and let α' be as in (1). Again, $\mathsf{v}_p(b') = k - 1 \geq 3$ and by induction hypothesis $[3, n-3] \subset \mathsf{L}(\alpha')$. Therefore $[4, n-2] \subset \mathsf{L}(\alpha)$ and by Claim D also $3 \in \mathsf{L}(\alpha)$.

If $k \geq n-1$, then the claim of the theorem follows from claims A, B, C, E and G. If $k \in [2, n-2]$, then the claim of the theorem follows from claims A, B and F. □

If $\alpha \in R(D)$ is a nonzero nonunit, and $\mathsf{L}(\alpha) = \{l_1, l_2, \ldots, l_k\}$, then the set of distances of α is defined as $\Delta(\alpha) = \{l_i - l_{i-1} \mid i \in [2, k]\}$, and $\Delta\big(R(D)\big) = \bigcup_{\alpha \in R(D) \setminus (\{0\} \cup U(R(D)))} \Delta(\alpha)$. For $k \in \mathbb{N}_{\geq 2}$, set $\mathcal{U}_k\big(R(D)\big) = \bigcup_{\alpha \in R(D), k \in \mathsf{L}(\alpha)} \mathsf{L}(\alpha)$.

Corollary 15. *If D is a PID, but not a field, then $\mathcal{U}_2\big(R(D)\big) = \mathbb{N}_{\geq 2}$ and $\Delta\big(R(D)\big) = \{1, 2\}$.*

Proof. This follows directly from Theorem 14. □

Corollary 16. *Suppose D is a PID that has infinitely many pairwise non-associated prime elements. Then*

$$\begin{aligned}\mathcal{L}(R(D)) = \Big\{ \{0\}, \{1\} \Big\} &\cup \Big\{ [m, n] \mid m \in [2, n] \Big\} \\ &\cup \Big\{ [m, n] \cup \{n+2\} \mid m \in [2, n] \text{ and } n \text{ even } \Big\} \\ &\cup \Big\{ [m, n] \cup \{n+2\} \mid m \in [3, n] \text{ and } n \text{ odd } \Big\} \\ &\cup \Big\{ m + 2[0, n] \mid \text{with } m \in \mathbb{N}_{\geq 2n} \text{ and } n \in \mathbb{N} \}.\end{aligned}$$

Proof. The sets $\{0\}$ and $\{1\}$ correspond to units and irreducibles. For zero-divisors, the sets of lengths are discrete intervals and completely described in Theorem 14(1). By Corollary 4 and Lemma 13(2), the sets of lengths of nonunit non-zero-divisors are arbitrary sumsets of sets as in Theorem 14(2), i.e., of sets of the form $\{1\}$, $[2, n]$ (for $n \geq 2$), $[3, n]$ (for $n \geq 3$), $[2, n] \cup \{n+2\}$ for even $n \geq 2$, and $[3, n] \cup \{n+2\}$ for odd $n \geq 3$. □

Finally, we remark that other important invariants of factorization theory (their definitions readily generalize to the zero-divisor case) are easily determined for $R(D)$ using the characterization of sets of lengths and Corollary 4.

Corollary 17. *Let D be a PID but not a field. $R(D)$ is a locally tame ring with catenary degree $\mathsf{c}\big(R(D)\big) = 4$. In particular, $\Delta\big(R(D)\big) = [1, \mathsf{c}\big(R(D)\big) - 2]$.*

Proof. We first observe that the catenary degree (see [8, Chapter 1.6] for the definition in the non-zero-divisor case) of $R(D)$ is 4: Let first $\alpha \in R(D)$ with $\mathrm{nr}(\alpha) \neq 0$. Using Corollary 4, we can reduce to the case where $\mathrm{nr}(\alpha)$ is a prime power. Since then $\min \mathsf{L}(\alpha) \le 3$, we can argue as in bifurcus semigroups (cf. [1, Theorem 1.1]), to find $\mathsf{c}(\alpha) \le 4$. In view of Lemma 8(1), and with a similar argument, the catenary degree of a zero-divisor is at most 2. Together this gives $\mathsf{c}(R(D)) \le 4$. Since there exists an element with set of lengths $\{2, 4\}$, also $\mathsf{c}(R(D)) \ge 4$.

We still have to show that $R(D)$ is locally tame (see [8, Chapter 1.6] or [10] for definitions). For this we have to show $\mathsf{t}(R(D), \gamma) < \infty$ for all irreducible $\gamma \in R(D)$. Let $\alpha \in R(D)$ and $\gamma \in R(D)$ be irreducible. If γ is prime, then $\mathsf{t}(R(D), \gamma) = 0$, hence we may suppose that γ is associated to one of the non-prime irreducibles from Theorem 5, and hence there exist a prime element $p \in D$ and $n \in \mathbb{N}$ such that $\mathrm{nr}(\gamma) = p^n$. If $\alpha \in R(D)$ is a zero-divisor, then $\mathsf{t}(\alpha, \gamma) = n$ follows easily from Lemma 8(1).

A standard technique allows us to show $\mathsf{t}(R(D)^\bullet, \gamma) < \infty$: By [10, Proposition 3.8], it suffices to show that two auxiliary invariants, $\omega(R(D)^\bullet, \gamma)$ and $\tau(R(D)^\bullet, \gamma)$ are finite.

Suppose $I \subset (R(D)^\bullet, \cdot)$ is a divisorial ideal. If we denote by ${}_{R(D)}\langle I \rangle$ the ideal of $R(D)$ generated by I, one checks that ${}_{R(D)}\langle I \rangle \cap R(D)^\bullet = I$. Since $R(D)$ is noetherian, $R(D)^\bullet$ is therefore v-noetherian. By [10, Theorem 4.2], $\omega(R(D)^\bullet, \gamma)$ is finite.

Recalling the definition of $\tau(\alpha, \gamma)$ (from [10, Definition 3.1]), it is immediate from Theorem 14 together with Corollary 4, that $\tau(R(D)^\bullet, \gamma) \le 3$. Altogether, therefore $\mathsf{t}(R(D), \gamma) < \infty$. $\square$

Remark 18. Suppose D is a PID but not a field.

(1) Trivially, Theorem 14(2) holds true for $R(D)^\bullet$.

(2) Let K be the quotient field of D, and $H = R(D)^\bullet$. We have

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, b \in D, a \in D^\bullet \right\},$$

and the complete integral closure of H is equal to

$$\widehat{H} = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, b \in K, a \in D^\bullet \right\}$$

because

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^n = \begin{pmatrix} a^n & na^{n-1}b \\ 0 & a^n \end{pmatrix}$$

for all $a, b \in K$ and $n \in \mathbb{N}$. This shows $H \neq \widehat{H}$, and even more we have $\mathfrak{f} = (H : \widehat{H}) = \emptyset$ (note that $(D : K) = \emptyset$). Thus the monoid under discussion is neither a Krull nor a C-monoid, which have been extensively studied in recent literature (see [8, Chapters 2.9, 3.3, and 4.6], [9], [12]).

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