Cyclically presented modules, projective covers and factorizations

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Abstract. We investigate projective covers of cyclically presented modules, characterizing the rings over which every cyclically presented module has a projective cover as the rings R that are Von Neumann regular modulo their Jacobson radical $J(R)$ and in which idempotents can be lifted modulo $J(R)$. Cyclically presented modules naturally appear in the study of factorizations of elements in non-necessarily commutative integral domains. One of the possible applications is to the modules M_R whose endomorphism ring $E := \mathrm{End}(M_R)$ is Von Neumann regular modulo $J(E)$ and in which idempotents lift modulo $J(E)$.

1. Introduction

An R-module M_R is said to be *cyclically presented* if $M_R \cong R/aR$ for some $a \in R$. In this paper, we study some natural connections between cyclically presented R-modules, their submodules, their projective covers and factorizations of elements in the ring R. That is, we find some results on projective covers of cyclically presented modules and apply them to the study of factorizations of elements in a ring. In this way, we are naturally led to the class of 2-firs. Recall that a ring R is a 2-fir if every right ideal of R generated by at most 2 elements is free of unique rank. This condition is right/left symmetric, and a ring R is a 2-fir if and only if it is a domain and the sum of any two principal right ideals with non-zero intersection is again a principal right ideal [**2**, Theorem 1.5.1]. P. M. Cohn investigated factorization of elements in 2-firs, applying the Artin-Schreier Theorem and the Jordan-Hölder-Theorem to the corresponding cyclically presented modules [**2**]. One of the main ideas developed in this paper is to characterize the submodules of a cyclically presented module M_R that, under a suitable cyclic presentation $\pi_M \colon R_R \to M_R$, lift to principal right ideals of R that are generated by a left cancellative element (Lemmas 2.2, 3.1 and 4.3). The key role is played by a class

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of cyclically presented submodules of a cyclically presented module M_R, which we call π_M*-exact submodules* of M_R. We show (Theorem 3.8) that, for every cyclically presented right R-module M_R and every cyclic presentation $\pi_M\colon R_R \to M_R$ with non-zero kernel, the set of all cyclically presented π_M-exact submodules is closed under finite sums if and only if R is a 2-fir. As we have said above, when sums and intersections of exact submodules are again exact submodules, we can use the Artin-Schreier and the Jordan-Hölder Theorems to study factorizations of elements.

We also study the rings over which every cyclically presented module has a projective cover. We characterize these rings as the rings R that are Von Neumann regular modulo their Jacobson radical $J(R)$ and in which idempotents can be lifted modulo $J(R)$ (Theorem 4.1). Finally, in the last Section, we consider the modules M_R whose endomorphism rings E are Von Neumann regular modulo the Jacobson radical $J(E)$ and in which idempotents can be lifted modulo $J(E)$. In particular, this applies to the case in which the module M_R in question is quasi-projective (Lemma 5.3 and Proposition 5.5).

Throughout the paper, R will be an associative ring with identity $1_R \neq 0_R$ and we will denote by $U(R)$ its group of invertible elements. By an R-module, we always mean a unitary right R-module.

2. Generalities

Let R be a ring. An element $a \in R$ is *left cancellative* if, for all $b, c \in R$, $ab = ac$ implies $b = c$. Equivalently, $a \in R$ is left cancellative if it is non-zero and is not a left zero-divisor. A (non-necessarily commutative) ring R is a *domain* if every non-zero element is left cancellative (equivalently, if every non-zero element is right cancellative). If $a \in R$, the right R-module homomorphism $\lambda_a\colon R_R \to aR, x \mapsto ax$, is an isomorphism if and only if a is left cancellative. More precisely, $aR \cong R_R$ if and only if there exists a left cancellative element $a' \in R$ with $a'R = aR$. If $a, a' \in R$ are two left cancellative elements, then $aR = a'R$ if and only if $a = a'\varepsilon$ for some $\varepsilon \in U(R)$.

Let $a, x_1, \dots, x_n \in R \setminus U(R)$ be $n+1$ left cancellative elements and assume that $a = x_1 \cdot \ldots \cdot x_n$. If $\varepsilon_1, \dots, \varepsilon_{n-1} \in U(R)$, then obviously also $a = (x_1\varepsilon_1) \cdot (\varepsilon_1^{-1} x_2 \varepsilon_2) \cdot \ldots \cdot (\varepsilon_{n-1}^{-1} x_n)$. This gives an equivalence relation on finite ordered sequences of left cancellative elements whose product is a. More precisely, if $F_a := \{\, (x_1, \dots, x_n) \mid n \geq 1,\ x_i \in R \setminus U(R)$ is left cancellative for every $i = 1, 2, \dots, n$ and $a = x_1 \cdot \ldots \cdot x_n \,\}$, then the equivalence relation $\sim$ on F_a is defined by $(x_1, \dots, x_n) \sim (x_1', \dots, x_m')$ if $n = m$ and there exist $\varepsilon_1, \dots, \varepsilon_{n-1} \in U(R)$ such that $x_1' = x_1\varepsilon_1$, $x_i' = \varepsilon_{i-1}^{-1} x_i \varepsilon_i$ for all $i = 2, \dots, n-1$ and $x_n' = \varepsilon_{n-1}^{-1} x_n$. In this paper, we call an equivalence class of F_a modulo $\sim$ a *factorization of a up to insertion of units*. Notice that the factors need not be irreducible. When this causes no confusion, we will simply call a representative of such an equivalence class a factorization.

A factorization $a = x_1 \cdot \ldots \cdot x_n$ gives rise to an ascending chain of principal right ideals, generated by left cancellative elements and containing aR:

$$aR \subsetneq x_1 \cdot \ldots \cdot x_{n-1}R \subsetneq \ldots \subsetneq x_1 R \subsetneq R,$$

hence to an ascending chain of cyclically presented submodules

$$0 = aR/aR \subsetneq x_1 \cdot \ldots \cdot x_{n-1}R/aR \subsetneq \ldots \subsetneq x_1R/aR \subsetneq R/aR$$

of the cyclically presented R-module R/aR. Notice that $x_1 \cdot \ldots \cdot x_{i-1}R/aR \cong R/x_i \cdot \ldots \cdot x_n R$ is cyclically presented because the elements x_i are left cancellative.

The next lemma shows that, conversely, every chain of principal right ideals generated by left cancellative elements in $aR \subset R$, determines a factorization of a into left cancellative elements, which is unique up to insertion of units.

LEMMA 2.1. *Let $a \in R$ be a left cancellative element, $aR = y_nR \subsetneq y_{n-1}R \subsetneq \dots \subsetneq y_1R \subsetneq y_0R = R$ be an ascending chain of principal right ideals of R, where $y_1, \dots, y_{n-1} \in R$ are left cancellative elements, $y_0 = 1$ and $y_n = a$. For every $i = 1, \dots, n$, let $x_i \in R$ be such that $y_{i-1}x_i = y_i$. Then $x_1, \dots, x_n$ are left cancellative elements and $a = x_1 \cdot \dots \cdot x_n$.*

Moreover, if $y'_1, \dots, y'_{n-1} \in R$ are also left cancellative elements with $y'_iR = y_iR$, $y'_0 = 1$ and $y'_n = a$, and we similarly define x'_i by $y'_{i-1}x'_i = y'_i$ for every $i = 1, 2, \dots, n$, then there exist $\varepsilon_1, \dots, \varepsilon_{n-1} \in U(R)$ such that $x'_1 = x_1\varepsilon_1$, $x'_i = \varepsilon_{i-1}^{-1}x_i\varepsilon_i$ for all $i = 2, \dots, n-1$ and $x'_n = \varepsilon_{n-1}^{-1}x_n$.

PROOF. Assume that x_i is not left cancellative for some $i = 1, 2, \dots, n$. Then there exists $b \neq 0$ such that $x_ib = 0$. Therefore $y_ib = y_{i-1}x_ib = 0$. This is a contradiction because y_i is left cancellative. Notice that $a = y_{n-1}x_n = y_{n-2}x_{n-1}x_n = \dots = y_0x_1 \dots x_n = x_1 \dots x_n$.

Now if $y'_iR = y_iR$ for every $i = 1, \dots, n-1$, then there exists $\varepsilon_1, \dots, \varepsilon_{n-1} \in U(R)$ such that $y'_i = y_i\varepsilon_i$. Therefore $y'_{i-1}x'_i = y_{i-1}x_i\varepsilon_i = y'_{i-1}\varepsilon_{i-1}^{-1}x_i\varepsilon_i$. But y'_{i-1} is left cancellative, so that $x'_i = \varepsilon_{i-1}^{-1}x_i\varepsilon_i$ for every $i = 2, \dots, n-1$.

Moreover, $y_1 = y_0x_1 = x_1$ and, similarly, $y'_1 = x'_1$, so that $y'_1 = y_1\varepsilon_1$ implies $x'_1 = x_1\varepsilon_1$. Finally, $y_{n-1}x_n = y_n = a = y'_n = y'_{n-1}x'_n = y_{n-1}\varepsilon_{n-1}x'_n$. Thus $x_n = \varepsilon_{n-1}x'_n$ and $x'_n = \varepsilon_{n-1}^{-1}x_n$. □

As we have already said in the introduction, we will characterize, in Lemmas 3.1 and 4.3, the submodules of cyclically presented modules M_R that, under a suitable cyclic presentation $\pi\colon R_R \to M_R$, that is, a suitable epimorphism $\pi\colon R_R \to M_R$, lift to principal right ideals of R generated by a left cancellative element. The following lemma will prove to be helpful to this end.

LEMMA 2.2. *Let A_R, B_R, M_R, N_R be modules over a ring R, $\pi_M\colon A_R \to M_R$ and $\pi_N\colon B_R \to N_R$ be epimorphisms, $\lambda\colon B_R \to A_R$ be a homomorphism and $\varepsilon\colon N_R \to M_R$ be a monomorphism such that $\pi_M\lambda = \varepsilon\pi_N$, so that there is a commutative diagram*

$$\begin{array}{ccc} B_R & \xrightarrow{\lambda} & A_R \\ {\scriptstyle \pi_N}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ N_R & \overset{\varepsilon}{\hookrightarrow} & M_R. \end{array}$$

Then the following three conditions are equivalent:

(a) $\pi_M^{-1}(\varepsilon(N_R)) = \lambda(B_R)$.
(b) $\lambda(\ker(\pi_N)) = \ker(\pi_M)$.
(c) π_M *induces an isomorphism* $\operatorname{coker}(\lambda) \to \operatorname{coker}(\varepsilon)$.

If, moreover, A'_R, B'_R are right R-modules such that there exist isomorphisms $\varphi_A\colon A'_R \to A_R$ and $\varphi_B\colon B'_R \to B_R$, and one defines $\pi'_N := \pi_N\varphi_B$, $\pi'_M := \pi_M\varphi_A$ and $\lambda' := \varphi_A^{-1}\lambda\varphi_B$, then the three conditions (a), (b) *and* (c) *are equivalent also to the the three conditions*

(d) $(\pi'_M)^{-1}(\varepsilon(N_R)) = \lambda'(B'_R)$.
(e) $\lambda'(\ker(\pi'_N)) = \ker(\pi'_M)$.
(f) π'_M *induces an isomorphism* $\operatorname{coker}(\lambda') \to \operatorname{coker}(\varepsilon)$.

Proof. (a) $\Leftrightarrow$ (b): We have $\pi_M\lambda(B_R) = \varepsilon\pi_N(B_R) = \varepsilon(N_R)$. It follows that $\pi_M^{-1}(\varepsilon(N_R)) = \lambda(B_R) + \ker\pi_M$. Thus (a) is equivalent to $\ker\pi_M \subseteq \lambda(B_R)$. The inclusion $\lambda(\ker(\pi_N)) \subseteq \ker(\pi_M)$ always holds by the commmutativity of the diagram, so that (b) is equivalent to $\ker(\pi_M) \subseteq \lambda(\ker(\pi_N))$. Thus (b) $\Rightarrow$ (a) is trivial. Conversely, if (a) holds, and $a \in \ker(\pi_M)$, then $a = \lambda(b)$ for some $b \in B_R$, so that $0 = \pi_M(a) = \pi_M\lambda(b) = \varepsilon\pi_N(b)$. But ε is mono, so $\pi_N(b) = 0$, and $a = \lambda(b) \in \lambda(\ker(\pi_N))$.

(b) $\Leftrightarrow$ (c) Apply the Snake Lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\pi_N) & \longrightarrow & B_R & \xrightarrow{\pi_N} & N_R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \lambda|_{\ker}} & & \downarrow{\scriptstyle \lambda} & & {\scriptstyle \varepsilon}\downarrow & & \\ 0 & \longrightarrow & \ker(\pi_M) & \longrightarrow & A_R & \xrightarrow[\pi_M]{} & M_R & \longrightarrow & 0, \end{array}$$

obtaining a short exact sequence

$$0 = \ker(\varepsilon) \longrightarrow \operatorname{coker}(\lambda|_{\ker}) \longrightarrow \operatorname{coker}(\lambda) \longrightarrow \operatorname{coker}(\varepsilon) \longrightarrow 0.$$

Therefore $\lambda(\ker(\pi_N)) = \ker(\pi_M)$ if and only if $\lambda|_{\ker}$ is surjective, if and only if $\operatorname{coker}(\lambda|_{\ker}) = 0$, if and only if the epimorphism $\operatorname{coker}(\lambda) \to \operatorname{coker}(\varepsilon)$ is injective, if and only if it is an isomorphism.

Now assume that there exist isomorphisms $\varphi_A\colon A'_R \to A_R$ and $\varphi_B\colon B'_R \to B_R$ and set $\pi'_N := \pi_N\varphi_B$, $\pi'_M := \pi_M\varphi_A$ and $\lambda' := \varphi_A^{-1}\lambda\varphi_B$. To conclude the proof, it suffices to show that $\lambda(\ker(\pi_N)) = \ker(\pi_M)$ if and only if $\lambda'(\ker(\pi'_N)) = \ker(\pi'_M)$. This is true, since $\ker(\pi'_M) = \varphi_A^{-1}(\ker(\pi_M))$ and

$$\lambda'(\ker(\pi'_N)) = \lambda'(\varphi_B^{-1}(\ker(\pi_N))) = \varphi_A^{-1}\lambda\varphi_B(\varphi_B^{-1}(\ker(\pi_N))) = \varphi_A^{-1}(\lambda(\ker(\pi_N))).$$

□

3. π-exactness

Let M_R be a cyclically presented right R-module and $\pi_M\colon R_R \to M_R$ a cyclic presentation. We introduce the notion of π_M-exactness to characterize those submodules of M_R that lift, via π_M, to principal right ideals of R, generated by a left cancellative element of R. We give sufficient conditions on R for this notion to be independent from the chosen presentation π_M.

Definition and Lemma 3.1 (π-exactness). *Let $N_R \leq M_R$ be cyclic right R-modules. Let $F_R \cong R_R$, fix an epimorphism $\pi_M\colon F_R \to M_R$ and let $\varepsilon\colon N_R \hookrightarrow M_R$ denote the embedding. The following conditions are equivalent:*

(a) $\pi_M^{-1}(N_R) \cong R_R$.

(b) *There exists a monomorphism $\lambda\colon R_R \to F_R$ and an epimorphism $\pi_N\colon R_R \to N_R$ such that $\lambda(\ker(\pi_N)) = \ker(\pi_M)$ and the following diagram commutes:*

$$\begin{array}{ccc} R_R & \xrightarrow{\lambda} & F_R \\ {\scriptstyle \pi_N}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ N_R & \xrightarrow[\varepsilon]{} & M_R. \end{array} \tag{3.1}$$

(c) *There exists a monomorphism* $\lambda\colon R_R \to F_R$ *and an epimorphism* $\pi_N\colon R_R \to N_R$ *such that diagram* (3.1) *commutes and induces an isomorphism* $\operatorname{coker}(\lambda) \to \operatorname{coker}(\varepsilon)$.

If these equivalent conditions are satisfied, we call N_R *a* π_M-exact submodule *of* M_R.

Proof. (a) $\Rightarrow$ (b). By (a), there exists an isomorphism $\lambda_0\colon R_R \to \pi_M^{-1}(N_R)$. Let λ be the composite mapping $R_R \xrightarrow{\lambda_0} \pi_M^{-1}(N_R) \hookrightarrow F_R$ and $\varepsilon^{-1}\colon \varepsilon(N_R) \to N_R$ be the inverse of the corestriction of ε to $\varepsilon(N_R)$. Noticing that $\pi_M\lambda(R_R) = \varepsilon(N_R)$, one gets an onto mapping $\pi_N := \varepsilon^{-1}\pi_M\lambda\colon R_R \to N_R$. Then diagram (3.1) clearly commutes and $\lambda(R_R) = \pi_M^{-1}(N_R)$. The statement now follows from Lemma 2.2.

(b) $\Leftrightarrow$ (c) and (b) $\Rightarrow$ (a). By Lemma 2.2. $\square$

Corollary 3.2. *Let* $F_R \cong R_R$ *and let* $\pi_M\colon F_R \to M_R$ *be an epimorphism. If* $\varphi\colon F'_R \to F_R$ *is an isomorphism and* $N_R \leq M_R$, *then* N_R *is a* π_M*-exact submodule of* M_R *if and only if it is a* $\pi_M\varphi$*-exact submodule of* M_R.

Proof. Let N_R be a π_M-exact submodule of M_R and let $\lambda\colon R_R \to F_R$ be a monomorphism satisfying condition (b) of Definition and Lemma 3.1. Apply Lemma 2.2 to $B_R = B'_R = R_R$, $A_R = F_R$, $A'_R = F'_R$, $\varphi_B = 1_R$ and $\varphi_A = \varphi$. Setting $\lambda' := \varphi^{-1}\lambda$, it follows that $\lambda'(\ker(\pi_N)) = \ker(\pi_M\varphi)$ and hence N_R is a $\pi_M\varphi$-exact submodule of M_R. The converse follows applying what we have just shown to φ^{-1}. $\square$

Corollary 3.3. *Let* $N_R \leq M_R$ *be cyclic* R*-modules,* $\pi_M\colon R_R \to M_R$ *be an epimorphism and* $N_R \leq M_R$ *be a* π_M*-exact submodule. Then* M_R/N_R *is cyclically presented with presentation induced by* π_M.

Proof. Let $\lambda\colon R_R \to R_R$ be as in condition (c) of Definition and Lemma 3.1. Then $M_R/N_R \cong R_R/\lambda(R_R)$, from which the conclusion follows immediately. $\square$

Corollary 3.4. *Let* $N_R \leq M_R \leq P_R$ *be cyclic* R*-modules and let* $\pi_P\colon F_R \to P_R$ *be an epimorphism, where* $F_R \cong R_R$. *If* $M_R \leq P_R$ *is* π_P*-exact and* $N_R \leq M_R$ *is* $\pi_P|_{\pi_P^{-1}(M_R)}$*-exact, then* $N_R \leq P_R$ *is* π_P*-exact.*

Proof. Set $F'_R := \pi_P^{-1}(M_R)$. By condition (a) of Definition and Lemma 3.1, $F'_R \cong R_R$. Therefore the notion of $\pi_P|_{F'_R}$-exactness of N_R in M_R is indeed defined. Since $\pi_P^{-1}(N_R) = (\pi_P|_{F'_R})^{-1}(N_R) \cong R_R$, the claim follows. $\square$

Let $c \in R$ be left cancellative and denote by $\mathsf{L}(cR, R)$ the set of all right ideals aR with $a \in R$ left cancellative and $cR \subset aR \subset R$. It is partially ordered by set inclusion. Let $\pi\colon R \to R/cR$ be an epimorphism. Denote by $\mathsf{L}_\pi(R/cR)$ the set of all π-exact submodules of R/cR. This set is also partially ordered by set inclusion.

Lemma 3.5. *Let* $c \in R$ *be left cancellative and let* $\pi\colon R_R \to R/cR$ *be the canonical epimorphism. Then* π *induces an isomorphism of partially ordered sets* $\mathsf{L}(cR, R) \cong \mathsf{L}_\pi(R/cR)$.

Proof. It suffices to show that $N_R \subset R/cR$ is π-exact if and only if there exists a left cancellative $a \in R$ with $\pi^{-1}(N_R) = aR$. But this is equivalent to $\pi^{-1}(N_R) \cong R_R$. The statement now follows from condition Definition and Lemma (a) of 3.1. $\square$

The following example shows that, in general, the condition of π-exactness indeed depends on the particular choice of the epimorphism $\pi\colon R_R \to M_R$. We refer the reader to any of [**5**], [**7**] or [**9**] for the necessary background on quaternion algebras.

Example 3.6. Let A be a quaternion algebra over $\mathbb{Q}$ and R be a maximal $\mathbb{Z}$-order in A such that there exists an unramified prime ideal $\mathfrak{P} \subset R$ and maximal right ideals I, J of R with $I, J \supset \mathfrak{P}$, I principal and J non-principal. Then $\mathfrak{p} = \mathfrak{P} \cap \mathbb{Z}$ is principal, say $\mathfrak{p} = p\mathbb{Z}$ with $p \in \mathbb{P}$, $\mathfrak{P} = pR$, $R/\mathfrak{P} \cong M_2(\mathbb{F}_p)$ and $\mathfrak{P} = \operatorname{Ann}(R/\mathfrak{P})$. (E.g., take $A = \left(\frac{-1,-11}{\mathbb{Q}}\right)$, $R = {}_{\mathbb{Z}}\langle 1, i, \frac{1}{2}(i+j), \frac{1}{2}(1+k)\rangle$, $p = 3$, $I = {}_{\mathbb{Z}}\langle \frac{1}{2}(1+5k), \frac{1}{2}(i+5j), 3j, 3k\rangle$ and $J = {}_{\mathbb{Z}}\langle \frac{1}{2}(1+2j+3k), \frac{1}{2}(i+3j+4k), 3j, 3k\rangle$).

The module $R/\mathfrak{P}$ has a composition series (as an $R/\mathfrak{P}$- and hence as an R-module)

$$0 \subsetneq I/\mathfrak{P} \subsetneq R/\mathfrak{P},$$

and there exists an isomorphism $R/\mathfrak{P} \to R/\mathfrak{P}$ mapping $J/\mathfrak{P}$ to $I/\mathfrak{P}$, as is easily seen from $R/\mathfrak{P} \cong M_2(\mathbb{F}_p)$. Therefore there exist epimorphisms $\pi_M\colon R \to R/\mathfrak{P}$ and $\pi'_M\colon R \to R/\mathfrak{P}$ with $\pi_M^{-1}(I/\mathfrak{P}) = I$ and $\pi_M'^{-1}(I/\mathfrak{P}) = J$. This implies that $I/\mathfrak{P}$ is a π_M-exact submodule of $R/\mathfrak{P}$ that is not π'_M-exact.

However, under an additional assumption on R_R, which holds, for instance, whenever R is a semilocal ring, the notion is independent of the choice of π.

Lemma 3.7. *Suppose that $R_R \oplus K_R \cong R_R \oplus R_R$ implies $K_R \cong R_R$ for all right ideals K_R of R.*

(1) *If $M_R \cong R/aR$ with $a \in R$ left cancellative and $\pi_M\colon R_R \to M_R$ is an epimorphism, then there exists a left cancellative $a' \in R$ such that $\ker(\pi_M) = a'R$.*
(2) *If M_R is a cyclic R-module, $\pi_M\colon R_R \to M_R$ and $\pi'_M\colon R_R \to M_R$ are epimorphisms and $N_R \le M_R$, then N_R is a π_M-exact submodule of M_R if and only if it is a π'_M-exact submodule of M_R.*

Proof. (1) Let $\pi_{aR}\colon R_R \to R/aR, 1 \mapsto 1 + aR$ be the canonical epimorphism. Since a is left cancellative, $aR \cong R_R$. Consider the exact sequences

$$0 \to aR \hookrightarrow R_R \xrightarrow{\pi_{aR}} R/aR \to 0$$

and

$$0 \to \ker(\pi_M) \hookrightarrow R_R \xrightarrow{\pi_M} R/aR \to 0.$$

By Schanuel's Lemma, $R_R \oplus aR \cong R_R \oplus \ker(\pi_M)$, and hence by assumption $aR \cong \ker(\pi_M)$. Thus there exists a left cancellative $a' \in R$ with $\ker(\pi_M) = a'R$.

(2) Let $\pi_{M/N}\colon M_R \to M_R/N_R$ be the canonical quotient module epimorphism. There are exact sequences

$$0 \to \pi_M^{-1}(N_R) \to R_R \xrightarrow{\pi_{M/N}\pi_M} M_R/N_R \to 0$$

and

$$0 \to \pi_M'^{-1}(N_R) \to R_R \xrightarrow{\pi_{M/N}\pi'_M} M_R/N_R \to 0,$$

and by Schanuel's Lemma therefore $R_R \oplus \pi_M^{-1}(N_R) \cong R_R \oplus \pi_M'^{-1}(N_R)$. If N_R is a π_M-exact submodule of M_R, then $\pi_M^{-1}(N_R) \cong R_R$ and hence $\pi_M'^{-1}(N_R) \cong R_R$ by our assumption on R, showing that N_R is a π'_M-exact submodule. The converse follows by symmetry. $\square$

Suppose that R has invariant basis number (for all $m, n \in \mathbb{N}_0$, $R_R^m \cong R_R^n$ implies $m = n$). Then the condition of the previous lemma is satisfied if every stably free R-module of rank 1 is free [**6**, §11.1.1]. This is true if R is commutative [**6**, §11.1.16]. The condition is also true if R is semilocal [**3**, Corollary 4.6] or R is a 2-fir (by [**2**, Theorem 1.1(e)]).

Let M_R be a right R-module with an epimorphism $\pi_M\colon R_R \to M_R$ with $\ker(\pi_M) = aR$ and $a \in R$ left cancellative. We say that a finite series

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M_R$$

of submodules is π_M-*exact*, if every M_i is an $\pi_M|_{\pi_M^{-1}(M_{i+1})}$-exact submodule of M_{i+1}. By Lemma 3.5 the π_M-exact series of submodules of R are in bijection with series of principal right ideals in $\mathsf{L}(aR, R)$. By Lemma 2.1 they are therefore in bijection with factorizations of a into left cancellative elements, up to insertion of units.

Recall that a ring R is a 2-fir if and only if it is a domain and the sum of any two principal right ideals with non-zero intersection is again a principal right ideal [**2**, Theorem 1.5.1]. In the next theorem, we will consider, for a cyclically presented right R-module M_R and a cyclic presentation $\pi_M\colon R_R \to M_R$ with non-zero kernel, the set of all submodules of cyclically presented π_M-exact submodules. We say it is *closed under finite sums* if for every two cyclically presented π_M-exact submodules M_1 and M_2 of M_R, the sum $M_1 + M_2$ also is cyclically presented and a π_M-exact submodule of M_R.

THEOREM 3.8. *Let R be a domain. The following are equivalent.*

(1) *For every cyclically presented right R-module M_R and every cyclic presentation $\pi_M\colon R_R \to M_R$ with non-zero kernel, the set of all cyclically presented π_M-exact submodules is closed under finite sums.*
(2) *R is a 2-fir.*

PROOF. (1) $\Rightarrow$ (2): Let $a, b, c \in R \setminus \{0\}$ be such that $cR \subset aR \cap bR$. We have to show that $aR + bR$ is right principal. Let $M_R = R/cR$, $\pi_M\colon R_R \to R/cR$ be the canonical epimorphism, $M_1 = aR/cR$ and $M_2 = bR/cR$. By Lemma 3.5, $M_1 = \pi_M(aR)$ and $M_2 = \pi_M(bR)$ are π_M-exact submodules of M_R. By assumption $M_1 + M_2$ is a π_M-exact submodule of M_R. Again by Lemma 3.5, $aR + bR = \pi_M^{-1}(M_1 + M_2)$ is a principal right ideal of R, generated by a left cancellative element.

(2) $\Rightarrow$ (1): We may assume $M_1, M_2 \neq 0$, as the statement is trivial otherwise. Let $\pi_M\colon R_R \to M_R$ be an epimorphism with non-zero kernel. Since M_1 and M_2 are π_M-exact submodules of M_R, there exist $a, b \in R \setminus \{0\}$ such that $\pi^{-1}(M_1) = aR$ and $\pi^{-1}(M_2) = bR$. Because $\ker(\pi) \neq 0$, we have $aR \cap bR \neq 0$. Since R is a 2-fir, there exists $c \in R \setminus \{0\}$ such that $aR + bR = \pi_M^{-1}(M_1 + M_2) = cR$. Therefore $M_1 + M_2$ is cyclically presented and a π_M-exact submodule of M_R. □

Notice that if we assume that sums and intersections of exact submodules are again exact submodules, one may use the Artin-Schreier and Jordan-Hölder-Theorems to study factorizations of elements. As we have just seen, such an assumption leads to the 2-firs investigated by Cohn in [**2**].

4. Projective covers of cyclically presented modules

Let R be a ring and R/xR a cyclically presented right R-module, $x \in R$. The module R/xR does not have a projective cover in general, but if it has one, it has one of the form $\pi|_{eR}\colon eR \to R/xR$, where $e \in R$ is an idempotent that depends on x and $\pi|_{eR}$ is the restriction to eR of the canonical projection $\pi\colon R_R \to R/xR$ [**1**, Lemma 17.17]. More precisely, given any projective cover $p\colon P_R \to R/xR$, there is an isomorphism $f\colon eR \to P_R$ such that $pf = \pi|_{eR}$. The kernel of the projective cover $\pi|_{eR}\colon eR \to R/xR$ is $eR \cap xR$ and is contained in $eJ(R)$ because the kernel of $\pi|_{eR}$ is a superfluous submodule of eR and $eJ(R)$ is the largest superfluous submodule of eR. Considering the exact sequences $0 \to xR \to R_R \to R/xR \to 0$ and $0 \to eR \cap xR \to eR \to R/xR \to 0$, one sees that $R_R \oplus (eR \cap xR) \cong eR \oplus xR$ (Schanuel's Lemma), so that $eR \cap xR$ can be generated with at most two elements.

Recall that every right R-module has a projective cover if and only if the ring R is perfect, and that every finitely generated right R-module has a projective cover if and only every simple right R-module has a projective cover, if and only if the ring R is semiperfect. Denoting by $J(R)$ the Jacobson radical of R, R is semiperfect if and only if $R/J(R)$ is semisimple and idempotents can be lifted modulo $J(R)$ [**1**, Theorem 27.6]. The next result gives a similar characterization for the rings R over which every cyclically presented right module has a projective cover.

Theorem 4.1. *The following conditions are equivalent for a ring R with Jacobson radical $J(R)$:*

(1) *Every cyclically presented right R-module has a projective cover.*

(2) *The ring $R/J(R)$ is Von Neumann regular and idempotents can be lifted modulo $J(R)$.*

Proof. Set $J := J(R)$.

(1) $\Rightarrow$ (2) Assume that every cyclically presented right R-module has a projective cover. In order to show that R/J is Von Neumann regular, it suffices to prove that every principal right ideal of R/J is a direct summand of the right R/J-module R/J [**4**, Theorem 1.1]. Let x be an element of R. We will show that $(xR + J)/J$ is a direct summand of R/J as a right R/J-module. By (1), the cyclically presented right R-module R/xR has a projective cover. As we have seen above, the projective cover is of the form $\pi|_{eR}\colon eR \to R/xR$ for some idempotent e of R, where $\pi\colon R_R \to R/xR$ is the canonical projection.

Applying the right exact functor $- \otimes_R R/J$ to the short exact sequence $0 \to eR \cap xR \to eR \to R/xR \to 0$, we get an exact sequence $(eR \cap xR) \otimes_R R/J \to eR \otimes_R R/J \to R/xR \otimes_R R/J \to 0$, which can be rewritten as $(eR \cap xR)/(eR \cap xR)J \to eR/eJ \to R/(xR + J) \to 0$. It follows that there is a short exact sequence $0 \to ((eR \cap xR) + eJ)/eJ \to eR/eJ \to R/(xR + J) \to 0$. Now the kernel $eR \cap xR$ of the projective cover $\pi|_{eR}$ is superfluous in eR and eJ is the largest superfluous submodule of eR, hence $((eR \cap xR) + eJ)/eJ = 0$ and $eR/eJ \cong R/(xR + J)$.

Now $(e + J)(R/J) = (eR + J)/J \cong eR/(eR \cap J) = eR/eJ$, so that $eR/eJ \cong R/(xR + J)$ is a projective right R/J-module. Thus the short exact sequence $0 \to (x + J)(R/J) = (xR + J)/J \to R/J \to R/(xR + J) \to 0$ splits, and the principal right ideal of R/J generated by $x + J$ is a direct summand of the right R/J-module R/J.

We must now prove that idempotents of R/J lift modulo J. By [**1**, Proposition 27.4], this is equivalent to showing that every direct summand of the R-module

R/J has a projective cover. Let M_R be a direct summand of $(R/J)_R$. Then it is also a direct summand of $(R/J)_{R/J}$ and hence is generated by an idempotent of R/J. Let $g \in R$ be such that $g+J \in R/J$ is idempotent and $M_{R/J} = (g+J)(R/J)$. Then $R/J = (g+J)(R/J) \oplus (1-g+J)(R/J)$ as R/J-modules, and hence also as R-modules. The canonical projection $\pi_g \colon R/J \to M_R$ has kernel $\ker(\pi_g) = (1-g+J)(R/J)$. Let $\pi\colon R_R \to R/J, r \mapsto r+J$ be the canonical epimorphism. Set $f := \pi_g \pi$. Then $\ker(f) = (1-g)R + J$ and so f factors through an epimorphism $\overline{f}\colon R/(1-g)R \to M_R$ with $\ker(\overline{f}) = (J + (1-g)R)/(1-g)R$. In particular, $\ker(\overline{f})$ is the image of the superfluous submodule J of R_R via the canonical projection $R_R \to R/(1-g)R$. It follows that $\ker(\overline{f})$ is superfluous in $R/(1-g)R$, i.e., $\overline{f}$ is a superfluous epimorphism.

By hypothesis, there is a projective cover $p\colon P_R \to R/(1-g)R$. Since the composite mapping of two superfluous epimorphisms is a superfluous epimorphism (this follows easily from [**1**, Corollary 5.15]), $\overline{f}p\colon P_R \to M_R$ is a superfluous epimorphism and hence a projective cover of M.

(2) $\Rightarrow$ (1) Assume that (2) holds. Let R/xR be a cyclically presented right R-module, where $x \in R$. The principal right ideal $(x+J)(R/J)$ of the Von Neumann regular ring R/J is generated by an idempotent and idempotents can be lifted modulo J. Hence there exists an idempotent element $e \in R$ such that $(x+J)(R/J) = (e+J)(R/J)$. Let $\pi|_{(1-e)R}$ be the restriction to $(1-e)R$ of the canonical epimorphism $\pi\colon R_R \to R/xR$. We claim that $\pi|_{(1-e)R}\colon (1-e)R \to R/xR$ is onto. To prove the claim, notice that $xR + J = eR + J$, so that $(1-e)R + xR + J = R$. As J is superfluous in R_R, it follows that $(1-e)R + xR = R$ and so $\pi|_{(1-e)R}$ is onto. This proves our claim. Finally, $\ker(\pi|_{(1-e)R}) = (1-e)R \cap xR \subseteq ((1-e)R + J) \cap (xR+J) = ((1-e)R + J) \cap (eR+J) \subseteq J$, so that $\ker(\pi|_{(1-e)R}) \subseteq J \cap (1-e)R = (1-e)J$ is superfluous in $(1-e)R$. Thus $\pi|_{(1-e)R}$ is the required projective cover of the cyclically presented R-module R/xR. □

Corollary 4.2. *If R is a domain and every cyclically presented right R-module has a projective cover, then R is local.*

Proof. By the previous Theorem, $R/J(R)$ is Von Neumann regular. Since idempotents lift modulo $J(R)$, the only idempotents of $R/J(R)$ are 0 and 1. Therefore $R/J(R)$ is a division ring and so R is local □

Notice that, conversely, if R is a local ring and M_R is any non-zero cyclic module, then every epimorphism $\pi\colon R_R \to M_R$ is a projective cover.

Lemma 4.3. *Let R be an arbitrary ring, let $N_R \leq M_R$ be cyclic right R-modules with a projective cover and let $\varepsilon\colon N_R \to M_R$ be the embedding. Then the following two conditions are equivalent:*

(1) *There exist a projective cover $\pi_N\colon P_R \to N_R$ of N_R, a projective cover $\pi_M\colon Q_R \to M_R$ of M_R and a commutative diagram of right R-module morphisms*

$$\begin{array}{ccc} P_R & \xrightarrow{\lambda} & Q_R \\ {\scriptstyle \pi_N}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ N_R & \xrightarrow[\varepsilon]{} & M_R, \end{array} \tag{4.1}$$

such that the following equivalent conditions hold:

(a) $\lambda(P_R) = \pi_M^{-1}(\varepsilon(N_R))$;
(b) $\lambda(\ker(\pi_N)) = \ker(\pi_M)$;
(c) π_M *induces an isomorphism* $\operatorname{coker}(\lambda) \to \operatorname{coker}(\varepsilon)$.

(2) *For every pair of projective covers* $\pi_N : P_R \to N_R$ *of* N_R *and* $\pi_M : Q_R \to M_R$ *of* M_R *and every commutative diagram* (4.1) *of right* R*-module morphisms, the following equivalent conditions hold:*
(a') $\lambda(P_R) = \pi_M^{-1}(\varepsilon(N_R))$;
(b') $\lambda(\ker(\pi_N)) = \ker(\pi_M)$;
(c') π_M *induces an isomorphism* $\operatorname{coker}(\lambda) \to \operatorname{coker}(\varepsilon)$.

Proof. The equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) and (a') $\Leftrightarrow$ (b') $\Leftrightarrow$ (c') have been proved in Lemma 2.2.

(b) $\Rightarrow$ (b'): Assume that $\pi_N : P_R \to N_R$, $\pi_M : Q_R \to M_R$ and $\lambda\colon P_R \to Q_R$ satisfy condition (b), that is, make diagram (4.1) commute and $\lambda(\ker(\pi_N)) = \ker(\pi_M)$. Let $\pi'_N : P'_R \to N_R$ and $\pi'_M : Q'_R \to M_R$ be projective covers and $\lambda' : P'_R \to Q'_R$ be a morphism that make the diagram corresponding to diagram (4.1) commute, that is, such that $\pi'_M\lambda' = \varepsilon\pi'_N$. Projective covers are unique up to isomorphism and, by Lemma 2.2, we may therefore assume $P'_R = P_R$, $Q'_R = Q_R$ and $\pi'_M = \pi_M$, $\pi'_N = \pi_N$.

Then $\pi_M(\lambda-\lambda') = \pi_M\lambda - \varepsilon\pi_N = \varepsilon\pi_N - \varepsilon\pi_N = 0$, so that $(\lambda-\lambda')(P_R) \subseteq \ker\pi_M$. Let $\iota\colon \ker\pi_M \to Q_R$ denote the inclusion. Then there exists a morphism $\psi\colon P_R \to \ker\pi_M$ such that $\lambda - \lambda' = \iota\psi$. As images via module morphisms of superfluous submodules are superfluous submodules and $\ker\pi_N$ is a superfluous submodule of P_R, it follows that $\psi(\ker\pi_N)$ is a superfluous submodule of $\ker\pi_M$. Now $\ker\pi_M = \lambda(\ker\pi_N) = (\lambda'+\iota\psi)(\ker\pi_N) \subseteq \lambda'(\ker\pi_N)+\iota\psi(\ker\pi_N) = \lambda'(\ker\pi_N)+\psi(\ker\pi_N) \subseteq \ker\pi_M$. Thus $\ker\pi_M = \lambda'(\ker\pi_N) + \psi(\ker\pi_N)$. But $\psi(\ker\pi_N)$ is superfluous in $\ker\pi_M$, hence $\ker\pi_M = \lambda'(\ker\pi_N)$, which proves (b').

(b') $\Rightarrow$ (b): Let $\pi_N : P_R \to N_R$ and $\pi_M : Q_R \to M_R$ be projective covers of N_R, respectively M_R. Since P_R is projective and $\pi_M : Q_R \to M$ is an epimorphism, there exists a $\lambda\colon P_R \to Q_R$ such that $\pi_M\lambda = \varepsilon\pi_N$. By (b'), then $\lambda(\ker(\pi_N)) = \ker(\pi_M)$. □

Definition 4.4. If $N_R \leq M_R$ are cyclic right R-modules and the equivalent conditions of Theorem 4.3 are satisfied, we say that N_R is an *exact submodule* of M_R.

Corollary 4.5. *If* $L_R \leq M_R \leq N_R$ *are cyclic right* R*-modules,* M_R *is exact in* N_R *and* L_R *is exact in* M_R*, then* L_R *is exact in* N_R.

Proof. Since L_R is exact in M_R and M_R is exact in N_R, there exist projective covers $\pi_L : P_R \to L_R$, $\pi_M : Q_R \to M_R$, $\pi'_M : Q'_R \to M_R$ and $\pi_N : U_R \to N_R$ and homomorphisms $\lambda\colon P_R \to Q_R$ and $\mu\colon Q'_R \to U_R$ such that $\pi_M\lambda = \pi_L$, $\pi_N\mu = \pi'_M$, $\lambda(\ker(\pi_L)) = \ker(\pi_M)$ and $\mu(\ker(\pi'_M)) = \ker(\pi_N)$.

Since the projective cover of M_R is unique up to isomorphism, we may assume by Lemma 2.2 that $Q_R = Q'_R$ and $\pi'_M = \pi_M$ (replacing λ accordingly). Then $\pi_N\mu\lambda = \pi_M\lambda = \pi_L$ and $\ker(\pi_N) = \mu(\ker(\pi_M)) = \mu(\lambda(\ker(\pi_L)) = (\mu\lambda)(\ker(\pi_L))$. Therefore N_R is an exact submodule of M_R. □

Corollary 4.6. *If a cyclic module* N_R *is an exact submodule of a cyclic module* M_R *and* M_R *has a projective cover isomorphic to* R_R*, then* M_R/N_R *is cyclically presented.*

Proof. Since N_R is an exact submodule of M_R, there exists a commutative diagram

$$\begin{array}{ccc} P_R & \xrightarrow{\lambda} & Q_R \\ {\scriptstyle \pi_N}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ N_R & \xrightarrow[\varepsilon]{} & M_R \end{array}$$

where $\pi_N\colon P_R \to N_R$ and $\pi_M\colon Q_R \to M_R$ are projective covers of N_R and M_R and $\operatorname{coker}(\lambda) \cong \operatorname{coker}(\varepsilon)$. By assumption, there exists an idempotent $e \in R$ such that $P_R \cong eR$ and $Q_R \cong R_R$. By Lemma 2.2, we may therefore assume $P_R = eR$ and $Q_R = R_R$ (replacing π_M, π_N and λ accordingly). Therefore $M_R/N_R = \operatorname{coker}(\varepsilon) \cong \operatorname{coker}(\lambda) = R/eR$. Hence M_R/N_R is cyclically presented. $\square$

The following example shows that if R is not a domain, then even if a non-unit $x \in R$ is not a zero-divisor, the projective cover of R/xR need not be isomorphic to R_R.

Example 4.7. Let D be a discrete valuation ring and $\pi \in D$ a prime element. The unique maximal ideal of D is πD. Let $R = M_2(D)$, $x = \begin{bmatrix} 1 & 0 \\ 0 & \pi \end{bmatrix}$ and $e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

We have

$$xR = \begin{bmatrix} D & D \\ \pi D & \pi D \end{bmatrix} \quad \text{and} \quad eR = \begin{bmatrix} 0 & 0 \\ D & D \end{bmatrix}.$$

Let $p\colon R_R \to R/xR$ be the canonical projection. We will show that $p|_{eR} : eR \to R/xR$ is a projective cover of R/xR. We have $\ker p|_{eR} = xR \cap eR = \begin{bmatrix} 0 & 0 \\ \pi D & \pi D \end{bmatrix}$. Since $J(R) = M_2(J(D)) = \begin{bmatrix} \pi D & \pi D \\ \pi D & \pi D \end{bmatrix}$, it follows that $\ker p|_{eR} = eJ(R)$. Since e is an idempotent of R, eR is projective and $eJ(R) = J(eR)$. In particular, $\ker p|_{eR}$ is superfluous in eR. Therefore eR is a projective cover of R/xR.

We now show that $eR \ncong R$. Assume eR is isomorphic to R. Then there exists an isomorphism $f\colon R_R \to eR$. Hence $f(1) = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \neq 0$.

Let $b = \begin{bmatrix} -d & 0 \\ c & 0 \end{bmatrix}$. Then $b \neq 0$, because $f(1) \neq 0$. But $f(1)b = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \begin{bmatrix} -d & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ implies $f(b) = 0$. It follows that $b = 0$, which contradicts $b \neq 0$. Thus eR is not isomorphic to R.

The next example shows that the condition for the projective cover of M_R to be isomorphic to R_R is necessary in Corollary 4.6.

Example 4.8. Let $R = T_2(\mathbb{Z}/2\mathbb{Z})$ be the ring of all upper triangular 2×2 matrices with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Since $J(R)$ consists of all strictly upper triangular matrices, $R/J(R) \cong (\mathbb{Z}/2\mathbb{Z})^2$ is semisimple and obviously idempotents lift modulo $J(R)$. Therefore every finitely generated R-module has a projective cover. Set

$$M_R := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\},$$

$$N_R := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} R = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\},$$

$$M_R/N_R = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + N_R, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + N_R \right\}.$$

Consider

$$\begin{array}{rcl} \phi\colon N_R & \longrightarrow & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} R \\ \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} & \longmapsto & \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \end{array}$$

It is obvious that ϕ is an isomorphism. Since $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is an idempotent of R, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} R$ is a projective R-module. Hence N_R is a projective R-module. On the other hand, M_R is also a projective R-module, because $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is an idempotent of R. Hence $1_N\colon N_R \to N_R$ and $1_M\colon M_R \to M_R$ are projective covers. This implies that the diagram

$$\begin{array}{ccc} N_R & \xrightarrow{\ \varepsilon\ } & M_R \\ {\scriptstyle 1_N}\downarrow & & \downarrow{\scriptstyle 1_M} \\ N_R & \xrightarrow[\ \varepsilon\ ]{} & M_R, \end{array}$$

where $\varepsilon(\ker 1_N) = \ker 1_M$, commutes. Therefore N_R is an exact submodule of M_R.

Assume M_R/N_R is a cyclically presented module. Then M_R/N_R is isomorphic to R/xR, where $x \in R$. Since $|M_R/N_R| = 2$, and $|xR| = 4$, we have:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} R = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\},$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} R = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} \right\} = N_R,$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R = M_R,$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} R = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & b+c \\ 0 & 0 \end{bmatrix} \right\} = M_R,$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} R = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \right\},$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} R = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & c \\ 0 & c \end{bmatrix} \right\},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R = R_R,$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} R = R_R.$$

Thus $xR = M_R$. Hence

$$R/xR = R/M_R = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + M_R, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + M_R \right\},$$

$$\begin{aligned} \operatorname{ann}(M_R/N_R) &= \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in R \;\middle|\; \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in N_R \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in R \;\middle|\; \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in N_R \right\} \\ &= \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}, \end{aligned}$$

$$\operatorname{ann}(R/xR) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in R \;\middle|\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in xR = M_R \right\} = M_R.$$

Hence $\operatorname{ann}(M_R/N_R) \neq \operatorname{ann}(R/xR)$. On the other hand, we have $\operatorname{ann}(M_R/N_R) = \operatorname{ann}(R/xR)$ since M_R/N_R is isomorphic to R/xR. This is a contradiction. Therefore M_R/N_R is not a cyclically presented module.

PROPOSITION 4.9. *Let R be a local domain. Let $N_R, M_R \neq 0$ be cyclically presented right R-modules and let $\pi_M \colon R_R \to M_R$ be an epimorphism. Then $N_R \subset M_R$ is exact if and only if it is π_M-exact in the sense of Definition and Lemma 3.1.*

PROOF. Suppose first $N_R \subset M_R$ exact. Let $\pi_N \colon R_R \to N_R$ be any epimorphism. Then π_M and π_N are necessarily projective covers, because $\ker(\pi_M)$ and $\ker(\pi_N)$ are superfluous. Let $\varepsilon \colon N_R \to M_R$ denote the inclusion. By projectivity of R_R, there exists a $\lambda \colon R_R \to R_R$ such that $\pi_M \lambda = \varepsilon \pi_N$. By condition (a) in Lemma 4.3, $\lambda(R_R) = \pi_M^{-1}(N_R)$. Since $\pi_M^{-1}(N_R) \neq 0$, it follows that $\pi_M^{-1}(N_R) \cong R_R$ and hence condition (a) in Definition and Lemma 3.1 is satisfied.

Suppose now that $N_R \subset M_R$ is π_M-exact. Let $\pi_N \colon R_R \to N_R$ be an epimorphism and $\lambda \colon R_R \to R_R$ a monomorphism satisfying condition (b) of Definition and Lemma 3.1. Then π_N is a projective cover of N_R, and condition (b) of Lemma 4.3 is satisfied, implying that $N_R \subset M_R$ is exact. □

The previous proposition, together with the results from the previous section, shows that in the special case of R a local domain and $x \in R$ a non-unit, series of exact submodules of R/xR may be used to study factorizations of $x \in R$ up to insertion of units.

5. Cokernels of endomorphisms

Let M_R be a right module over a ring R and let $E := \operatorname{End}(M_R)$ be its endomorphism ring. Let s be a fixed element of E. In this section, we investigate the relation between projective covers $eE \to E/sE$ for an idempotent e, induced by the canonical epimorphism $E_E \to E/sE$, and properties of the module $e(M_R)$. This is of particular interest if we assume that $E/J(E)$ is Von Neumann regular and idempotents can be lifted modulo $J(E)$, as in this case for every non-zero $s \in E$ the module E/sE has a projective cover. For instance, every continuous module M_R has this property [**8**, Proposition 3.5 and Corollary 3.9], in particular every quasi-injective module has this property, and every module of Goldie dimension one and dual Goldie dimension one has this property [**8**, Proposition 2.5].

Let $s\colon M_R \to M_R$ be an endomorphism of M_R. We can consider the direct summands M_1 of M_R such that there exists a direct sum decomposition $M_R = M_1 \oplus M_2$ of M_R for some complement M_2 of M_1 with the property that $\pi_2 s\colon M_R \to M_2$ is a split epimorphism. Here $\pi_2\colon M_R \to M_2$ is the canonical projection with kernel M_1. Let $\mathcal{F}$ be the set of all such direct summands, that is,

$$\mathcal{F} := \{\, M_1 \mid M_1 \leq M_R, \ \text{ there exists } M_2 \leq M_R \text{ such that } M_R = M_1 \oplus M_2 \\ \text{and } \pi_2 s\colon M_R \to M_2 \ \text{ a split epimorphism}\,\}.$$

The set $\mathcal{F}$ can be partially ordered by set inclusion.

It is well known that there is a one-to-one correspondence between the set of all pairs (M_1, M_2) of R-submodules of M_R such that $M_R = M_1 \oplus M_2$ and the set of all idempotents $e \in E$. If $e \in E$ is an idempotent, the corresponding pair is the pair $(M_1 := e(M_R), M_2 := (1-e)(M_R))$. If $s \in \operatorname{End}(M_R)$, we always denote by $\varphi\colon E_E \to E/sE$ the canonical epimorphism $\varphi(f) = f + sE$.

Lemma 5.1. *Let $M_R = M_1 \oplus M_2$, let $\pi_2\colon M_R \to M_2$ be the projection with kernel M_1, and let $e \in \operatorname{End}(M_R)$ be the endomorphism corresponding to the pair (M_1, M_2). If $s\colon M_R \to M_R$ is an endomorphism, then $\pi_2 s$ is a split epimorphism if and only if $\varphi|_{eE}\colon eE \to E/sE$ is surjective.*

Proof. We have to show that $\pi_2 s\colon M_R \to M_2$ is a split epimorphism if and only if $eE + sE = E$. In order to prove the claim, assume that $\pi_2 s\colon M_R \to M_2$ is a split epimorphism, so that there is an R-module morphism $f\colon M_2 \to M_R$ with $\pi_2 s f = 1_{M_2}$. Let $\varepsilon_2\colon M_2 \to M_R$ be the embedding. Then the right ideal $eE + sE$ of E contains the endomorphism

$$\begin{aligned} e(1_M - sf\pi_2) + s(f\pi_2) &= e + (1_M - e)sf\pi_2 = e + \varepsilon_2\pi_2 sf\pi_2 \\ &= e + \varepsilon_2 1_{M_2}\pi_2 = e + (1_M - e) = 1_M, \end{aligned}$$

so that $eE + sE = E$. Conversely, let $e \in E$ be an idempotent with $eE + sE = E$, so that there exist $g, h \in E$ with $1 = eg + sh$. Then $(1-e) = (1-e)sh$, so that $(1-e) = (1-e)sh(1-e)$, that is, $\varepsilon_2\pi_2 = \varepsilon_2\pi_2 sh\varepsilon_2\pi_2$. Since ε_2 is injective and π_2 is surjective, they can be canceled, so that $1_{M_2} = \pi_2 sh\varepsilon_2$. Hence $\pi_2 s$ is a split epimorphism, which proves our claim. □

Proposition 5.2. *Let M_R be a right module, and let $E := \operatorname{End}(M_R)$ be its endomorphism ring. Let $s \in E$ and suppose that E/sE has a projective cover. Then*

$$\mathcal{F} := \{\, M_1 \mid M_1 \leq M_R, \ \textit{ there exists } M_2 \leq M_R \textit{ such that } M_R = M_1 \oplus M_2 \\ \textit{and } \pi_2 s\colon M_R \to M_2 \ \textit{ a split epimorphism}\,\}$$

has minimal elements, and all minimal elements of $\mathcal{F}$ are isomorphic R-submodules of M_R.

Proof. From the previous lemma, it follows that there is a one-to-one correspondence between the set $\mathcal{F}'$ of all pairs (M_1, M_2) of R-submodules of M_R such that $M_R = M_1 \oplus M_2$ and $\pi_2 s\colon M_R \to M_2$ is a split epimorphism and the set of all idempotents $e \in E$ for which the canonical mapping $eE \to E_E/sE$, $x \in eE \mapsto x + sE$, is surjective. In order to prove that $\mathcal{F}$ has minimal elements, it suffices to show that if the canonical mapping $eE \to E_E/sE$ is a projective cover, then $e(M_R)$ is a minimal element of $\mathcal{F}$. Let $e \in E$ be such that $eE \to E_E/sE$ is a projective cover, and let $M_1' \in \mathcal{F}$ be such that $M_1' \subseteq e(M_R)$. Let $e' \in E$ be an idempotent such that $M_1' = e'(M_R)$ and $\pi_2' s\colon M_R \to (1-e')(M_R)$ is a split epimorphism. Then $M_1' = e'(M_R) \subseteq e(M_R)$, so that $ee' = e'$. Thus $e'E = ee'E \subseteq eE$. If $\varphi|_{eE}\colon eE \to E/sE$ is the projective cover, $\varphi|_{e'E}\colon e'E \to E/sE$ denotes the canonical epimorphism and $\varepsilon\colon e'E \to eE$ is the embedding, it follows that $\varphi|_{eE}\varepsilon = \varphi|_{e'E}$. Now $\varphi|_{eE}$ is a superfluous epimorphism and $\varphi|_{eE}\varepsilon = \varphi|_{e'E}$ is onto, so that ε is onto, that is, $e'E = eE$. Thus $e = e'f$ for some $f \in E$, so that $e(M_R) \subseteq e'(M_R) = M_1'$ and $M_1' = e(M_R)$. It follows that $e(M_R)$ is a minimal element of $\mathcal{F}$.

Now let M_1'' be any other minimal element of $\mathcal{F}$, and let e'' be an idempotent element of E with $\pi_2'' s\colon M_R \to (1-e'')(M_R)$ a split epimorphism. Then the canonical projection $e''E \to E/sE$ is an epimorphism. As the canonical projection $\varphi|_{eE}\colon eE \to E/sE$ is the projective cover, there is a direct sum decomposition $e''E = P_E' \oplus P_E''$ with the canonical projection $P_E' \to E/sE$ a projective cover. Thus $P_E' = p'E$ for some idempotent p' of E with $p'E + sE = E$, so that $p'(M_R) \in \mathcal{F}$. Now $e''E \supseteq P_E' = p'E$ implies that $p' = e''g$ for some $g \in E$, so that $p'(M_R) \subseteq e''(M_R) = M_1''$. By the minimality of M_1'' in $\mathcal{F}$, it follows that $p'(M_R) = e''(M_R)$, so that $M_1'' = e''(M_R) = p'(M_R) \cong p'E \otimes_E M_R = P' \otimes_E M_R \cong eE \otimes_E M_R \cong e(M_R)$. Thus every minimal element of $\mathcal{F}$ is isomorphic to $e(M_R)$. $\square$

We conclude the paper by considering quasi-projective modules. Let M_R and N_R be right R-modules. Recall that M_R is *N_R-projective* if for every epimorphism $f\colon N_R \to L_R$ and every homomorphism $g\colon M_R \to L_R$ there exists a homomorphism $h\colon M_R \to N_R$ such that $g = fh$. Equivalently, for every epimorphism $f\colon N_R \to L_R$, the induced homomorphism $f_*\colon \operatorname{Hom}_R(M_R, N_R) \to \operatorname{Hom}_R(M_R, L_R)$ is surjective. If M_R is N_R-projective and $K_R \leq N_R$, then M_R is also K_R-projective [**1**, Proposition 16.12(1)]. A right R-module M_R is *quasi-projective* if it is M_R-projective. Trivially, projective modules and semisimple modules are quasi-projective.

Let M_R be quasi-projective, $E := \operatorname{End}_R(M_R)$ and suppose $s \in E$. In the following, we relate projective covers of the R-module $M_R/s(M_R)$ and the cyclically presented E-module E/sE.

LEMMA 5.3. *Let M_R be a quasi-projective right R-module, E the endomorphism ring of M_R and let $s \in E$. Let π be the canonical epimorphism of M_R onto $M_R/s(M_R)$ and φ the canonical epimorphism of E_E onto E/sE.*

(1) *For every $g \in E$, $\pi|_{g(M_R)}$ is surjective if and only if $\varphi|_{gE}$ is surjective.*
(2) *For every $g \in E$, gE is a direct summand of E_E if and only if $g(M_R)$ is a direct summand of M_R.*
(3) *Let e, e' be idempotents in E. Then $e(M_R) \cong e'(M_R)$ if and only if $eE \cong e'E$.*
(4) *Let $e \in E$ be idempotent. Then $\ker(\pi|_{e(M_R)})$ is superfluous if and only if $\ker(\varphi|_{eE})$ is superfluous.*

PROOF. (1) ($\Leftarrow$) Since $\varphi|_{gE}$ is surjective, there exists h in E such that $gh + sE = 1_M + sE$. Hence there exists h' in E such that $gh = 1_M + sh'$. For all $m \in M_R$ we have $\pi(m) = \pi(1_M(m)) = \pi(g(h(m)))$, whence $\pi|_{g(M_R)}$ is surjective.

($\Rightarrow$) Since M_R is quasi-projective and $\pi g\colon M_R \to M_R$ is an epimorphism, there exists $h\colon M_R \to M_R$ such that $\pi g h = \pi$. Therefore $(gh - 1_M)(M_R) \subset s(M_R)$. Since $s\colon M_R \to s(M_R)$ is an epimorphism, quasi-projectivity of M_R implies that there exists $h' \in E$ such that $gh - 1_M = sh'$. This implies that $\varphi(gh) = 1_M + sE$. Therefore $\varphi|_{gE}$ is surjective.

(2) ($\Rightarrow$) If gE is a direct summand of E, there exists an idempotent e in E such that $gE = eE$. Hence there exist h, h' in E such that $g = eh$ and $e = gh'$. This implies that $g(M_R) = e(M_R)$. On the other hand, $e(M_R)$ is a direct summand of M_R since e is an idempotent of E. Therefore $g(M_R)$ is a direct summand of M_R.

($\Leftarrow$) If $g(M_R)$ is a direct summand of E, there exists an idempotent e in E such that $g(M_R) = e(M_R)$. Hence $eg = g$. Therefore $gE \subset eE$. Since $g\colon M_R \to e(M_R)$ is an epimorphism and M_R is quasi-projective, there exists $h\colon M_R \to M_R$ such that $e = gh$. This implies that $eE \subset gE$. Hence $eE = gE$.

(3) ($\Leftarrow$) Since $eE \cong e'E$, there exists an isomorphism $\Gamma\colon eE \to e'E$. Consider the two following homomorphisms $f\colon e(M_R) \to e'(M_R)$ defined via $f(m) = e'x(m)$ where $e'x = \Gamma(e)$ and $g\colon e'(M_R) \to e(M_R)$ defined via $g(m) = ey(m)$ where $ey = \Gamma^{-1}(e')$. It suffices to show that $fg = 1_{e'(M_R)}$ and $gf = 1_{e(M_R)}$. For $m \in e'(M_R)$, $fg(m) = f(ey(m)) = e'xey(m) = e'xy(m) = \Gamma(e)y(m) = \Gamma(ey)(m) = \Gamma(\Gamma^{-1}(e'))(m) = e'(m) = m$, it follows that $fg = 1_{e'(M_R)}$. By an argument analogous to the previous one, we get $gf = 1_{e(M_R)}$.

($\Rightarrow$) Since $e(M_R) \cong e'(M_R)$, there exists an isomorphism $h\colon e(M_R) \to e'(M_R)$. Consider the two following homomorphisms $\theta\colon eE \to e'E$ defined via $\theta(ex) = e'hex$, and $\theta'\colon e'E \to eE$ defined via $\theta'(e'x) = eh^{-1}e'x$. It suffices to show that $\theta\theta' = 1_{e'E}$ and $\theta'\theta = 1_{eE}$. Since $\theta\theta'(e'x)(m) = \theta(eh^{-1}e'x)(m) = e'heh^{-1}e'x(m) = e'he(h^{-1}(e'x(m))) = e'h(h^{-1}(e'x(m))) = e'e'(x(m)) = e'(x(m))$, it follows that $\theta\theta'(e'x) = e'x$. Hence $\theta\theta' = 1_{e'E}$. By an argument analogous to the previous one, we get $\theta'\theta = 1_{eE}$.

(4) ($\Rightarrow$) Let K_E be a submodule of eE such that $K_E + \ker(\varphi|_{eE}) = eE$. It suffices to show that $K_E = eE$. There exists $h \in \ker(\varphi|_{eE}) = eE \cap sE$ and $k \in K_E$ such that $e = k + h$. Hence $e(M_R) = k(M_R) + h(M_R)$. This implies that $e(M_R) = k(M_R) + \big(e(M_R) \cap s(M_R)\big)$. Since $e(M_R) \cap s(M_R)$ is superfluous in $e(M_R)$, then $e(M_R) = k(M_R)$. Since $k\colon M_R \to e(M_R)$ is an epimorphism and M_R is quasi-projective, there exists h' in E such that $e = kh'$. This implies that $e \in K_E$. Therefore $K_E = eE$.

($\Leftarrow$) Let N_R be a submodule of M_R such that $N_R + \ker(\pi|_{e(M_R)}) = M_R$. Hence $\pi|_{N_R}$ is surjective. It suffices to show that $N_R = M_R$. Since M_R is quasi-projective and N_R is a submodule of M_R, it follows that M_R is also N_R-projective. Therefore the induced homomorphism $(\pi|_{N_R})_*\colon \mathrm{Hom}(M_R, N_R) \to \mathrm{Hom}(M_R, M_R/s(M_R))$ is surjective and hence there exists $g\colon M_R \to N_R$ such that $\pi g = \pi e$. Again by quasi-projectivity of M_R, there exists $h\colon M_R \to M_R$ such that $g - e = sh$. Since $g(M_R) \subset N_R \subset e(M_R)$, for every $x \in M_R$ there exists $y \in M_R$ such that $g(x) = e(y)$. We have $eg(x) = e(e(y)) = e(y) = g(x)$. Thus $eg = g$. Since $g - e = eg - e = sh$, $eg - e \in eE$ and $sh \in sE$, it follows that $g - e \in eE \cap sE$. From $e = g - (g - e)$, we have $eE = gE + (g-e)E$. Hence $eE = gE + (eE \cap sE)$. Since $eE \cap sE = \ker \varphi|_{eE}$ is superfluous, $eE = gE$. Therefore $e(M_R) = g(M_R) \subset N_R$. Thus $N_R = e(M_R)$. □

Corollary 5.4. *Let M_R be a projective right R-module and E the endomorphism ring of M_R. Let $s \in E$ and let π be the canonical epimorphism from M_R to $M_R/s(M_R)$ and φ the canonical epimorphism from E to E/sE. Then $\pi|_{e(M_R)}$ is a projective cover of $M_R/s(M_R)$ if and only if $\varphi|_{eE}$ is a projective cover of E/sE.*

Proof. Since M_R is projective, so is $e(M_R)$. Hence $\pi|_{e(M_R)}$ is a projective cover if and only if $\ker(\pi|_{e(M_R)})$ is superfluous. Therefore the corollary follows from the previous lemma. □

PROPOSITION 5.5. *Let M_R be a quasi-projective right R-module, let $s \in E = \operatorname{End}(M_R)$ and let $\pi\colon M_R \to M_R/s(M_R)$ be the canonical epimorphism. Suppose that E/sE has a projective cover.*

Consider $\mathcal{E} := \{ N_R \le M_R \mid \pi|_{N_R}$ is surjective$\}$ and $\mathcal{E}_\oplus := \{ N_R \in \mathcal{E} \mid N_R$ is a direct summand of $M_R\}$, both partially ordered by set inclusion. Then $\mathcal{E}_\oplus$ has minimal elements, any two minimal elements of $\mathcal{E}_\oplus$ are isomorphic as right R-modules and any minimal element of $\mathcal{E}_\oplus$ is minimal in $\mathcal{E}$.

PROOF. Let $N_R \le M_R$ be a direct summand of M_R, let $e \in E$ be an idempotent with $e(M_R) = N_R$ and let $\pi_2\colon M_R \to \ker(e)$ be the canonical projection corresponding to the direct sum decomposition $M_R = N_R \oplus \ker(e)$. Lemma 5.3(1) implies that $\pi|_{N_R}\colon N_R \to M_R/s(M_R)$ is surjective if and only if $\varphi|_{eE}\colon eE \to E/sE$ is surjective. By Lemma 5.1 this is the case if and only if $\pi_2 s$ is a split epimorphism. This shows that $\mathcal{E}_\oplus = \mathcal{F}$, where the latter is defined as in Proposition 5.2. The claims about $\mathcal{E}_\oplus$ therefore follow from the proposition.

It remains to show that the minimal elements of $\mathcal{E}_\oplus$ are minimal in $\mathcal{E}$. Let $N_R \in \mathcal{E}_\oplus$ be minimal, and let $e\colon M_R \to N_R$ be an idempotent with $e(M_R) = N_R$. From the proof of Proposition 5.2, we see that $eE \to E/sE$ is a projective cover. Therefore Lemma 5.3(4) implies that $\ker(\pi|_{N_R})$ is superfluous. Therefore, if $L_R \le N_R$ and $\pi|_{L_R}$ is surjective, we have $L_R + \ker(\pi|_{N_R}) = N_R$ and hence $L_R = N_R$, showing that N_R is minimal in $\mathcal{E}$. □

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