Course Notes

Grundthema Algebra: Ring Theory

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for PhD students, 3h/week

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Ring Theory

I Rings of Quotients

(Lam, LMR, Chapter 4 and Chapter 3)

1. Issues with nc localization

Commutative localization: R comm. ring, $S \subseteq R$ multiplicative set ($0 \notin S$, $1 \in S$, $S \cdot S \subseteq S$)

$\Longrightarrow \exists$ ring $S^{-1}R$ & hom $\varepsilon: R \to S^{-1}R$ such that $\varepsilon(S) \subseteq (S^{-1}R)^{\times}$ and the following universal property (UP) holds:

(UP) For every ring hom $\varphi: R \to T$ for which $\varphi(S) \subseteq T^{\times}$, $\exists!$ hom $\overline{\varphi}: S^{-1}R \to T$ s.t. $\overline{\varphi} \circ \varepsilon = \varphi$

$$\begin{tikzcd} R \arrow[r, "\varepsilon"] \arrow[dr, "\varphi"'] & S^{-1}R \arrow[d, dashed, "\exists!\, \overline{\varphi}"] \\ & T \end{tikzcd}$$

$S^{-1}R$ also has the following properties:

(a) $S^{-1}R = \{ \varepsilon(a)\varepsilon(s)^{-1} : a \in R, s \in S \}$,

(b) $\ker(\varepsilon) = \{ r \in R : \exists s \in S : rs = 0 \}$.

$S^{-1}R$ is the localization of R at S.

In the nc setting several problems arise!

Say $\varphi: R \to T$ is S-inverting ($S \subseteq R$ multiplicative) if $\varphi(S) \subseteq T^{\times}$

Prop 1.1 („The Good") Let R be a (nc) ring and $S \subseteq R$ a multiplicative set. Then there exists a ring R_S and a S-inverting hom $\varepsilon: R \to R_S$ satisfying the following UP: for all S-inverting hom's $\varphi: R \to T$, there exists a unique $\overline{\varphi}: R_S \to T$ s.t. $\varphi = \overline{\varphi} \circ \varepsilon$.

<u>Proof:</u> Fix a presentation $\pi : \mathbb{Z}\langle X \rangle \twoheadrightarrow R$ ($\ker \pi = \langle G \rangle$)

generators ... gen. by relations

For each $s \in S$ pick $\tilde{s} \in \pi^{-1}(s)$, and add a new generator s^* and relations $s^*\tilde{s} = 1$, $\tilde{s}s^* = 1$.

$X' := X \cup \{s^* : s \in S\}$, $I := \langle G, s^*\tilde{s} - 1, \tilde{s}s^* - 1 : s \in S \rangle$

Define $R_S := \mathbb{Z}\langle X' \rangle / I$. Then there is a hom $\varepsilon : R \to R_S$, by construction $\varepsilon(S) \subseteq R_S^\times$, and ε satisfies the UP. $\square$

But what does R_S look like? (a), (b)?

<u>Example:</u> •) k nonzero ring, $R := M_n(k)$, $n \geq 2$.

$S := \{1, E_{11}\}$

Let $\varepsilon : R \to R_S$. $\ker(\varepsilon)$ is an ideal of R, so $\ker(\varepsilon) = M_n(I)$ for an ideal I of k.

$E_{11}E_{22} = 0 \Rightarrow E_{22} \in \ker(\varepsilon) = M_n(I) \Rightarrow 1 \in I \Rightarrow I = k \Rightarrow \ker(\varepsilon) = R$.

<u>so:</u> $R_S = 0$ and $\varepsilon = 0$!

•) There even exist <u>domains</u> R with $R_S = 0$ (Exc. 5, Ch. 4.9., LMR)

<u>So:</u> R_S unpredictable; as (a), (b) can fail

•) no good description of $\ker(\varepsilon)$

•) elements of R_S i.g. are sums of words in $\varepsilon(r)$, $\varepsilon(s)^{-1}$, e.g. expressions like

$$\varepsilon(r_1)\varepsilon(s_1)^{-1}\varepsilon(r_2)\varepsilon(s_2)^{-1} + \varepsilon(r_1')\varepsilon(s_1')^{-1}\varepsilon(r_2)$$

may not simplify further.

"The Bad": • Not every domain D can be embedded in a division ring

• Not every cancellative monoid H can be embedded in a group (Mal'cev 1937)

Approach: first construct suitable H, then show $\mathbb{Q}H$ is a domain

Lemma 1.2 Let H be a semigroup, $a, b, c, d, x, y, u, v \in H$. If $H \subseteq G$ for a group G, then:

$$ax = by, \quad cx = dy, \quad au = bv \quad \Rightarrow \quad cu = dv \qquad (\text{in } H)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & u \\ -y & -v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \boxed{*} \end{pmatrix}$$

Proof: $b^{-1}a = yx^{-1} = \boxed{d^{-1}c}$

$$b^{-1}a = \boxed{vu^{-1}}$$

So $d^{-1}c = vu^{-1} \Rightarrow cu = dv$. $\square$

Theorem 1.3 There exists a cancellative monoid H with elements a, b, c, d, x, y, u, v s.t. $ax = by$, $cx = dy$, $au = bv$ but $cu \neq dv$. In particular, H cannot be embedded into a group.

Proof: Let $\hat{H} = \{A, B, C, D, X, Y, U, V\}^*$ (free monoid)

For $W, W' \in \hat{H}$ let $W \sim W'$

$\Leftrightarrow$ W can be transformed into W' by a sequence of replacements of the form

$$AX \leftrightarrow BY, \quad CX \leftrightarrow DY, \quad AU \leftrightarrow BV.$$

Then $\sim$ is a congruence relation, the set of $\sim$-equivalence classes forms a monoid H. Notation: $w := [W]_\sim$.

In H: $ax = by$, $cx = dy$, $au = bv$

but $\boxed{cu \neq dv}$ $\quad ([CU]_\sim = \{CU\},\ [DV]_\sim = \{DV\})$

Suffices to show: H is cancellative

normal form: Say $W \in \hat{H}$ is reduced if it does not contain AX, CX, AU.

Every word $W \in \hat{H}$ is equivalent to a unique reduced word $(AX \to \underline{B}Y,\ CX \to \underline{D}Y,\ AU \to \underline{B}V)$

To prove cancellativity, we look at products of reduced words (and reduce the product ...)

Show: $ww_1 = ww_2 \in H \implies w_1 = w_2$ in H

Say w, w_1, w_2 are repr. by reduced words W, W_1, W_2

Case 1: WW_1, WW_2 are reduced $\implies WW_1 = WW_2$ (uniqueness)

$\implies W_1 = W_2$ ($\hat{H}$ cancellative) $\implies w_1 = w_2$

Case 2: (w.l.o.g.) WW_1 not reduced

Several cases, we look at $W = \dots L\underline{\underline{A}}$, $W_1 = \underline{\underline{X}}M_1N_1\dots$

$\implies ww_1$ repr. by reduced word $\dots L(B\underline{Y})M_1N_1\dots$

Case 2a: W_2 does not start with X or U

$\implies WW_2$ reduced, but $WW_2 = \dots L\boxed{A}W_2$
$\neq \dots L(\boxed{B}Y)M_1$ $\lightning$

Case 2b: $W_2 = U\dots$

$\implies WW_2 \sim \dots L(BV)\dots$ $\lightning$ $V \neq Y$

Case 2c: $W_2 = XM_2N_2\dots$

$\implies WW_2 \sim \dots LBYM_2N_2\dots$ (reduced)

$\implies M_1N_1\dots = M_2N_2\dots$

$\implies W_1 = W_2 \implies w_1 = w_2$

(Other cases, right cancellation similar) $\square$

<u>Theorem 1.4</u> Let k be a domain and H as in Thm 1.3. Then the semigroup algebra $R = kH$ is a domain and R cannot be embedded into any division ring.

<u>Proof</u>: Suffices to show that R is a domain. (If $R \hookrightarrow D$, D div. ring, then $H \hookrightarrow D^\times \leftarrow$ group, which is impossible by Thm 1.3)

Suppose
$$0 = \Big(\sum_i{}' \alpha_i w_i\Big)\Big(\sum_j{}' \alpha_j' w_j'\Big) \in R \quad (*)$$
with $\alpha_i, \alpha_j' \neq 0$, w_i $[w_j']$ given by distinct reduced words
$$W_i \quad [W_j']$$
Assume w.l.o.g. all W_i, W_j' have same length (pass to terms of longest lengths)

For $(*)$ to hold, must have $W_1 W_1' = W_i W_j'$ for some $i \neq 1$, $j \neq 1$.

$W_1 \neq W_i$, $|W_1| = |W_i| \Rightarrow$

(Say)
$$W_1 = \ldots LA \qquad W_1' = XMN\ldots$$
$$W_i = \ldots LB \qquad W_j' = YMN\ldots$$
But then the term $\alpha_1 \alpha_j' w_1 w_j'$, will reduced word $\ldots L\underline{AY}MN\ldots$ cannot be cancelled by any other term

Other cases are similar: $\begin{matrix} C-X \\ D-Y \end{matrix} \rightsquigarrow CY$

$\begin{matrix} A-U \\ B-V \end{matrix} \rightsquigarrow AV$

$\square$

<u>Remark</u>: A better way: H can be <u>ordered</u>, i.e., $\exists$ total order $<$ on H s.t.
$$\alpha < \beta \Rightarrow \alpha\gamma < \beta\gamma \ \wedge\ \gamma\alpha < \gamma\beta \quad \forall \alpha, \beta, \gamma \in H$$
Then looking at minimal elements in the support shows kH is a domain. <u>Ordering</u> First by length, then lexicographical
$v > a > b > y > c > d > u > x$.

<u>More difficult</u>: Domain R s.t. $R \setminus \{0\}$ embeds into a group, but R does not embed into a division ring.

ring theoretic obstructions, e.g. R must have IBN.

<u>Leavitt algebras</u>: $n > m \geq 1$ $R = R_{m,n}$ $2mn$ gen's

$$\{ x_{ij}, y_{ke} : 1 \leq i, e \leq m, \ 1 \leq j, k \leq n \}$$

<u>relations</u>: $(x_{ij})(y_{ke}) = I_m$, $(y_{ke})(x_{ij}) = I_n$

$\Longrightarrow$ $R^m_R \cong R^n_R$, so R doesn't have IBN.

$\Longrightarrow$ no hom to a ring with IBN.

Cohn ('66): $R_{m,n}$ is a domain if $m > 1$.

<u>„The Ugly"</u>: Sometimes A can be embedded into a division ring, but there is no unique „minimal" choice.

(*)

<u>Recall</u>: If R is a ring, $\sigma \in End(R)$, the <u>skew polynomial ring</u> $R[x;\sigma]$ has R-basis $\{x^i : i \geq 0\}$ and multiplication defined by $xr = \sigma(r)x$.
Elements can be (uniquely) written in the form

$$\sum_{i=0}^{m} r_i x^i \quad , r_i \in R \qquad \text{(coefficients on the } \underline{\underline{\text{left}}}\text{)}$$

- If R is a domain & σ is injective, $R[x;\sigma]$ is a domain [Look at leading terms]
- If R is a division ring & σ is injective $R[x;\sigma]$ is a PLID (= principal left ideal domain), in particular left noetherian.
  [Polynomial division & Euclidean algorithm]

(*) A division ring D is a division hull of A if $\exists$ embedding $j: A \to D$ s.t. $j(A)$ generates D [i.e. $\nexists$ div. ring D_0 : $j(A) \subseteq D_0 \subsetneq D$]
division-hulls D, D' are „the same" if they are A-isomorphic:

$$\begin{tikzcd} & D \arrow[d, "\varphi", "\wr"'] \\ A \arrow[ur, "j"] \arrow[dr, "j'"'] \arrow[r, phantom, "\circlearrowleft"] & D' \end{tikzcd}$$

We will show: If $\mathbb{C}$ is a field, the free algebra $C\langle u,v\rangle$ has infinitely many pairwise different division hulls.

Thm 1.5: Every right noetherian domain can be embedded into a division ring

(Proof: Later – using Goldie's Thm.)

To embed $C\langle u,v\rangle$ into division rings, embed it into suitable skew polynomial rings.

Lemma 1.6 Let k be a ring, $\sigma \in \mathrm{End}(k)$ injective, $R = k[x;\sigma]$.
If $\{t_i : i \in I\} \subseteq k$ are right linearly independent over $\sigma(k)$, then $\{t_i x : i \in I\} \subseteq R_R$ are right linearly independent over R.

Proof: Suppose $0 = \sum_{i \in I} (t_i x) f_i$, $f_i \in R$, a.a. $f_i = 0$.

Write $f_i = \sum_j a_{ij} x^j$, $a_{ij} \in k$.

$$\Rightarrow 0 = \sum_{i \in I} t_i x \sum_j a_{ij} x^j = \sum_j \Big(\sum_{i \in I} t_i \sigma(a_{ij}) \Big) x^{j+1}$$

$$\Rightarrow \forall j: 0 = \sum_{i \in I} t_i \sigma(a_{ij}) \Rightarrow \forall i \in I\ \forall j: \sigma(a_{ij}) = 0 \Rightarrow \forall i\ \forall j: a_{ij} = 0.$$

$$\Rightarrow \forall i: f_i = 0 \qquad \square$$

Lemma 1.7 (Sategaonkar's Lemma)

Let R be a ring, $a, b \in R$ right linearly independent over R. Let $0 \neq C \subseteq R$ be a subring whose elements commute with a, b. Then the subring of R gen. by C, a, b is a free C-ring on a, b.
C commuting C-algebra

C commutes with x, y

Proof: Suppose not. Let $f \in C\langle x,y\rangle$ be a nonzero polynomial of minimal degree s.t. $f(a,b)=0$, $n = \deg f$.

Let $f(x,y) = \alpha + x\,g(x,y) + y\,h(x,y) \quad (\alpha \in C)$

Without restriction, $\boxed{g \neq 0}$ (otherwise swap a, b)

$$0 = f(a,b)\,b = a\,\underline{g(a,b)\,b} + b(\alpha + h(a,b)\,b)$$

$$\overset{a,b \text{ lin indep}}{\Longrightarrow} 0 = g(a,b)\,b$$

$$g(x,y) = \beta + x\,p(x,y) + y\,q(x,y) \quad (\beta \in C)$$

$$\Longrightarrow \deg g \leq n-1, \quad \deg p, \deg q \leq n-2$$

$$0 = g(a,b)\,b = \underline{a}(p(a,b)\,b) + \underline{b}(\beta + q(a,b)\,b)$$

$$\Longrightarrow 0 = p(a,b)\,b = \beta + q(a,b)\,b$$

$$\Longrightarrow p(x,y)\,y, \quad \beta + q(x,y)\,y \quad \text{vanish on } a, b$$

$$\overset{\text{degree}}{\Longrightarrow} 0 = p(x,y) = \beta = q(x,y) \Rightarrow g(x,y) = 0 \quad ⚡. \quad \square$$

Now: C a field, $k := C(t)$ rational function field, $n > 1$.

$\sigma_n : k \to k, \quad \sigma_n|_C = id, \quad \sigma_n(t) = t^n$

(σ_n is not surjective, $\sigma_n(k) = C(t^n) \subsetneq C(t)$)

$R_n := k[x; \sigma_n]$

$\{1, t\}$ lin. indep $/ C(t^n) = \sigma_n(k) \overset{1.6}{\Longrightarrow} \{x, tx\}$ R_n-linearly indep.

$$\overset{1.7}{\Longrightarrow} C\langle u,v\rangle \hookrightarrow R_n, \quad u \mapsto x, \; v \mapsto tx.$$

C-embedding

By (1.5), R_n embeds into a div hull D_n, giving an embedding $\varepsilon_n: \mathbb{C}\langle u,v\rangle \hookrightarrow D_n$, $u \mapsto x$, $v \mapsto tx$

<u>Thm 1.8</u> (1) D_n is a division hull for $\mathbb{C}\langle u,v\rangle$

(2) For $n \neq m$, $n,m > 1$, $\nexists$ ring hom $f: D_m \to D_n$ with $f \circ \varepsilon_m = \varepsilon_n$ (i.e. D_m, D_n are essentially different div. hulls of $\mathbb{C}\langle u,v\rangle$)

<u>Proof</u>: (1) Suppose $\operatorname{im}(\varepsilon_n) \subseteq E \subseteq D_n$ with E a div. ring.

$\Rightarrow x, tx \in E \Rightarrow t = (tx)x^{-1} \in E \Rightarrow \mathbb{C}(t) = k \subseteq E$

$\Rightarrow k[x; \delta_n] = R_n \subseteq E \underset{\substack{\uparrow \\ D_n \text{ div. hull of } R_n}}{\Rightarrow} E = D_n$.

(2) Suppose such f exists.

$$\left(\varepsilon_m(u)^{-1}\varepsilon_m(v)\right)^m = (x^{-1}tx)^m = x^{-1}t^m x \overset{\overset{R_m}{\downarrow}}{=} x^{-1}xt = t = \underbrace{\varepsilon_m(v)}_{tx}\varepsilon_m(u)^{-1} \in D_m$$

and $\left(\varepsilon_n(u)^{-1}\varepsilon_n(v)\right)^n = \varepsilon_n(v)\,\varepsilon_n(u)^{-1} \in D_n$

Apply f to get:

$$\underbrace{\left(\varepsilon_n(u)^{-1}\varepsilon_n(v)\right)^m}_{= t^m} = \underbrace{\left(\varepsilon_n(u)^{-1}\varepsilon_n(v)\right)^n}_{= t^n}$$

$\Rightarrow t^{m-n} = 1$ ↯ $m \neq n$.

<u>Remark</u>: If $F = \mathbb{C}\langle z_1, z_2, \dots\rangle$ is free in (at most) countably many variables, then $F \hookrightarrow \mathbb{C}\langle u,v\rangle$ by $z_i \mapsto uv^i$, and F embeds in a division ring.

More general one can show: If D is any div. ring, X any set, then $D\langle X\rangle$ embeds into a div. ring. (FC-(14.25), different method – Malcev-Neumann construction)

2. Classical Rings of Quotients

2.1. Ore Localizations

Let R be a ring, $S \subseteq R$ a multiplicative set.

Def 2.1 A ring R' is a <u>right ring of fractions</u> (or <u>right quotient ring</u>) w.r.t. $S \subseteq R$ if there is a (fixed) ring hom. $\varphi: R \to R'$ s.t.

(i) φ is S-inverting

(ii) $R' = \{ \varphi(r)\varphi(s)^{-1} : r \in R, s \in S \}$

(iii) $\ker \varphi = \{ r \in R : \exists s \in S : rs = 0 \}$

<u>Note</u>: if $R \neq 0$, then $R' \neq 0$ by (iii)

<u>Necessary conditions for such an R' to exist:</u>

(1) Consider $\varphi(s)^{-1}\varphi(r)$ $(r \in R, s \in S)$:

By (ii) $\exists r' \in R \ \exists s' \in S$: $\varphi(s)^{-1}\varphi(r) = \varphi(r')\varphi(s')^{-1}$

$$\Rightarrow \varphi(rs') = \varphi(sr') \Rightarrow rs' - sr' \in \ker\varphi$$

$$\overset{(iii)}{\Rightarrow} \exists s'' \in S: \quad r\underbrace{s's''}_{\in S} - s\underbrace{r's''}_{\in R} = 0$$

$$\Rightarrow \boxed{rS \cap sR \neq \emptyset \quad \forall r \in R \ \forall s \in S} \quad (*)$$

If $(*)$ holds, S is a <u>right Ore set</u> (or <u>right permutable</u>)

(2) Suppose $sr = 0$ for some $s \in S$, $r \in R$.

$$\Rightarrow 0 = \varphi(sr) = \varphi(s)\varphi(r) \overset{(i)}{\Rightarrow} \varphi(r) = 0 \overset{(iii)}{\Rightarrow} \exists s' \in S: \ \underline{rs' = 0}$$

<u>So</u>: $\boxed{\forall s \in S \ \forall r \in R: \quad sr = 0 \Rightarrow \exists s' \in S: \ rs' = 0}$

S is <u>right rerersible</u>

Def 2.2 If S is right Ore & right reversible, then S is a <u>right denominator set</u>.

Theorem 2.3 (Ore, Asano) R has a right ring of fractions wrt. S $\Leftrightarrow$ S is a right denominator set.

Proof: „$\Rightarrow$": See discussion before Def. 2.2.

„$\Leftarrow$" On $R \times S$ define $\sim$ by

$$(a,s) \sim (a',s') \Leftrightarrow \exists b,b' \in R: \; sb = s'b' \in S \text{ and } ab = a'b' \; (\in R)$$

[„ $as^{-1} = ab(\overset{S}{\overset{\Downarrow}{sb}})^{-1} = a'b'(s'b')^{-1} = a'(s')^{-1}$"] ($b, b'$ need not be in S)

$\sim$ is an equivalence relation: reflexive & symmetric ✓

transitive: let $(a,s) \sim (a',s')$, $(a',s') \sim (a'',s'')$

say $sb = \underset{\in S}{s'b'}$, $ab = a'b'$ $\quad$ $s'c = \underset{\in S}{s''c'}$, $a'c = a''c'$ $\quad$ $(c,c',b,b' \in R)$

(Ore) $\underbrace{s'c}_{\in S} S \cap s'b'R \neq \emptyset \Rightarrow \exists r \in R \; \exists t \in S: \; s'ct = s'b'r \in S$

$\overset{\text{right reversibility}}{\Longrightarrow} \exists t' \in S: \; ctt' = b'rt'$.

So $sbr = s'b'r = s'ct = s''c't \in S$

$\overset{\cdot t'}{\Longrightarrow} s(brt') = s''(c'tt')$

and $a(brt') = a'b'rt' = a'ctt' = a''(c'tt')$
$\qquad\qquad\qquad\quad$ ↑ right rev.

$\Rightarrow (a,s) \sim (a'',s'')$

Write as^{-1} for the equivalence class of (a,s), and let RS^{-1} be the set of equivalence classes. Note: $(a,s) \sim (ab, sb)$ if $sb \in S$

Addition: let $a_1 s_1^{-1}, a_2 s_2^{-1} \in RS^{-1}$

$\Rightarrow s_1 S \cap s_2 R \neq \emptyset \Rightarrow \exists r \in R, s \in S: \; s_1 s = s_2 r \in S$ (Common denominator)

$$\Rightarrow a_1 s_1^{-1} + a_2 s_2^{-1} = a_1 s(\underbrace{s_1 s}_{=:t})^{-1} + a_2 r(\underbrace{s_2 r}_{=:t})^{-1} = (a_1 s + a_2 r)t^{-1}$$

Then $(RS^{-1}, +)$ is a group w. neutral element $0 \cdot 1^{-1}$,
$\varphi: R \to RS^{-1}$, $a \mapsto a1^{-1}$ is a hom.
$\ker(\varphi) = \{ a \in R : (a,1) \sim (0,1) \} = \{ a \in R : \exists s \in S : as = 0 \}$

<u>Multiplication</u>: $a_1 s_1^{-1}, a_2 s_2^{-1} \in RS^{-1}$

$s_1 R \cap a_2 S \neq \emptyset \Rightarrow \exists r \in R, s \in S: \quad s_1 r = a_2 s$

So "$s_1^{-1} a_2 = r s^{-1}$"

So <u>define</u> $(a_1 s_1^{-1}) \cdot (a_2 s_2^{-1}) = \underbrace{(a_1 r)(s_2 s)^{-1}}_{\in RS^{-1}}$

Now check (i) RS^{-1} is a ring, (tedious)
(ii) φ is an S-inverting ring hom (✓)

$\square$

<u>Cor. 2.4</u> If S is a right denominator set, then
$\varphi: R \to RS^{-1}$, $r \mapsto r1^{-1}$ is a universal S-inverting hom.
In <u>particular</u>: $\exists g: R_S \xrightarrow{\sim} RS^{-1}$ s.t. $g \circ \varepsilon = \varphi$ $\quad (\varepsilon: R \to R_S)$

<u>Proof</u>: Let $\alpha: R \to T$ be S-inverting.
Define $f: RS^{-1} \to T$ by $f(as^{-1}) = f(a)\underbrace{f(s)^{-1}}_{\in T^\times}$ $\quad (a \in R, s \in S)$
<u>Well-defined</u> Suppose $b \in R$, $sb \in S$

$\Rightarrow \underbrace{\alpha(sb)}_{\in T^\times} = \underbrace{\alpha(s)}_{\in T^\times} \alpha(b) \Rightarrow \alpha(b) \in T^\times$

$\Rightarrow \alpha((ab)(sb)^{-1}) = \alpha(a) \cancel{\alpha(b)} \cancel{\alpha(b)^{-1}} \alpha(s)^{-1} = \alpha(a)\alpha(s)^{-1}$

<u>Ring hom</u>: check ✓

<u>$f \circ \varphi = \alpha$</u> ✓

<u>Uniqueness</u> ✓

$\square$

<u>Cor 2.5</u> If S is a right denominator set & left denominator set, then RS^{-1} and $S^{-1}R$ exist, and $RS^{-1} \cong S^{-1}R$ (canonically)

<u>Rem/Exc</u>: (1) If $S \subseteq R$ is a right Ore set, then
$$(\text{ass } S) = I := \{ a \in R : as = 0 \text{ for some } s \in S \}$$
is an ideal of R

(2) Let $S \subseteq R$ be a right denominator set, $\overline{R} := R/I$
$\Rightarrow \overline{S} \subseteq \overline{R}$ is a right denominator set consisting of regular elements and $RS^{-1} \cong \overline{R}\,\overline{S}^{-1}$ (as R-rings).

<u>Examples 2.6</u> (Nonexamples)

(1) If k is a domain, $|X| \geq 2$, $k\langle X\rangle$ is not Ore:
If $a, b \in X$ w. $a \neq b$, then $a\,k\langle X\rangle \cap b\,k\langle X\rangle = 0$

(2) $R = \mathbb{Z}\langle x, y\rangle / \langle yx = 0,\ y^2 = 0\rangle$
$\Longrightarrow R = \{ f(x) + g(x)y : f, g \in \mathbb{Z}[x] \}$
$S := \{ x^n : n \geq 0 \}$

- S contains <u>no left zero divisors</u> $[\,x^n(f(x) + g(x)y) = 0 \Rightarrow f(x) = g(x) = 0\,]$
- S is <u>right Ore</u>: Let $a = f(x) + g(x)y \in R$, $s = x^n$ $(n \geq 0)$
<u>Claim</u>: $aS \cap sR \neq \emptyset$
Wlog $n \geq 1$
$$\Rightarrow a\underbrace{x^n}_{\in S} = f(x)x^n = x^n f(x) = s\underbrace{f(x)}_{\in R}$$
$\Longrightarrow RS^{-1}$ exists

S is <u>not</u> <u>left Ore</u>: (Find $a \in R, s \in S$: $Rs \cap Sa = \emptyset$)

$(a = y,\ s = x)$

$\forall n: 0 \neq \underbrace{x^n}_{\in S} \underbrace{y}_{=a} \implies Sy \not\subseteq \{0\}$

$Rx = \{ f(x)x + \underbrace{g(x)yx}_{=0} : f, g \in \mathbb{Z}[x] \} = x\,\mathbb{Z}[x]$

$\implies Sy \cap Rx = \emptyset$.

$\implies S^{-1}R$ does <u>not</u> <u>exist</u>

<u>What is RS^{-1}?</u> $y \in I \overset{\text{Exc}}{\implies} I = \langle y \rangle$

$\bar{R} := R/\langle y \rangle \cong \mathbb{Z}[x]$, $\bar{S} = \{x^n : n \geq 0\} \implies RS^{-1} = \bar{R}\bar{S}^{-1} = \mathbb{Z}[x^{\pm 1}]$

Laurent polynomials.

<u>2.2 Right Ore rings & domains</u>

Let R be a ring, $S \subseteq R$ a multiplicative subset

<u>Remark</u> <u>1.</u> If S is central, it is Ore. So RS^{-1}, $S^{-1}R$ exist, $S^{-1}R \cong RS^{-1}$ and $RS^{-1} \cong R \otimes_{Z(R)} Z(R)S^{-1}$

<u>2.</u> Say $s \in R$ is <u>regular</u> if it is neither left nor right zero-divisor ($\Leftrightarrow$ s is cancellative)
If S consists of reg. elements, then S is left & right reversible.

<u>3.</u> The regular elements form a multiplicative subset $S = R^\circ$ of R.

<u>Def. 2.7</u> Let $S = R^\circ$.

(1) R is a <u>right Ore ring</u> if S satisfies the right Ore condition ($\Leftrightarrow$ RS^{-1} exists).
Then RS^{-1} is the <u>(total) classical ring of quotients of R</u>. <u>Notation</u>: $Q^r_{cl}(R)$.

(2) If R is left & right Ore, then we say R is an Ore ring, and $Q^r_{cl}(R) = Q^e_{cr}(R)$.

Remark: (1) If R is commutative, it is an Ore ring.

(2) Let R be a domain, $S = R^\circ = R\setminus\{0\}$.

Then R right Ore $\Leftrightarrow$ $aR \cap bR \neq \{0\}$ for $a, b \in R\setminus\{0\}$.

right Ore condition

Thm 2.8 (by Goldie) For a domain R, TFAE

(a) R is right Ore

(b) R_R is uniform ($\Leftrightarrow \forall\, 0 \neq A, B \leq R_R:\ A \cap B \neq 0$
$\Leftrightarrow$ R_R does not contain a nontrivial direct sum of submodules
$\Leftrightarrow$ $\operatorname{udim} R_R = 1$.)

(c) R_R does not contain an infinite direct sum of submodules ($\Leftrightarrow \operatorname{udim} R_R < \infty$)

Proof: (a) $\Rightarrow$ (b) Suppose $A, B \neq 0$. Let $a \in A\setminus\{0\}$, $b \in B\setminus\{0\}$
$\overset{\text{r. Ore}}{\Longrightarrow}$ $aR \cap bR \neq 0 \Rightarrow A \cap B \neq 0$.

(b) $\Rightarrow$ (a) Apply (2) w. $A = aR$, $B = bR$

(b) $\to$ (c) ✓

(c) $\to$ (a) Suppose $a, b \in R^\circ$ are such that $aR \cap bR = 0$.
We show $\{a^i b : i \geq 0\}$ are right R-linearly independent.

Suppose $\sum_{i=m}^{n} a^i b r_i = 0$ with $0 \leq m \leq n$, $r_i \in R$, $r_m \neq 0$

and $n-m$ minimal.

$$a^m(br_m + abr_{m+1} + \dots + a^{n-m} b r_{n-m}) = 0$$

$$\Rightarrow br_m + abr_{m+1} + \dots + a^{n-m} br_{n-m} = 0$$

$$\Rightarrow 0 \neq br_m = a(-b_{r_{m+1}} - \dots - a^{n-m-1} br_{n-m}) \in bR \cap aR \quad \text{↯}$$

$$\Rightarrow \bigoplus_{i \geq 0} a^i bR \leq R \qquad \square$$

$\uparrow$ free

<u>Note</u>: Thm 2.8 is false for non-domains, e.g. If $D_1, \ldots, D_m$ are div. rings, then $R = D_1 \times \ldots \times D_m$ is Ore, but $\operatorname{udim}(D_1 \times \ldots \times D_m) = m$.

<u>Cor 2.9</u> Let R be a domain

(1) If R is right noetherian, then R is Ore.
Then $Q^r_{cl}(R)$ is the unique division hull of R.

(2) If R is a right Bézout domain (every f.g. right ideal is principal), then R is Ore

(3) If R is <u>not</u> Ore, then R contains a free algebra $C\langle x_1, x_2, \ldots \rangle$ with $C = Z(R)$.

<u>Proof</u>: (1) Suppose not. By Thm 2.8(3), $\exists$ nonzero modules $A_1, A_2, \ldots \subseteq R_R$ s.t. $\bigoplus_{i \geq 1} A_i \subseteq R_R$. But $A_1 \subsetneq A_1 \oplus A_2 \subsetneq A_1 \oplus A_2 \oplus A_3 \subsetneq \ldots$ is an infinite ascending chain of submodules $\lightning$.

(2) Suppose $a, b \in R^\bullet$ s.t. $aR \cap bR = 0$. Choose $c \in R$ s.t. $cR = aR + bR = aR \oplus bR$.
$\Rightarrow c = ar + bs$ with $r, s \in R$.
$bR \subseteq cR \Rightarrow b = cd$ with $d \in R \setminus \{0\}$
$\Rightarrow b = cd = ard + bsd \Rightarrow ard \in aR \cap bR \Rightarrow rd = 0$
$\Rightarrow r = 0 \Rightarrow c = bs \Rightarrow cR = bR$ $\lightning$ $a \neq 0$.

(3) Suppose R is not right Ore. $\Rightarrow \exists a, b \in R$ s.t. a, b are right linearly independent
$\overset{\text{Jategaonkar's Lemma 1.7}}{\Longrightarrow}$ $C\langle a, b \rangle$ is free

$C\langle x_1, x_2, x_3, \ldots \rangle \hookrightarrow C\langle a, b \rangle$, $x_i \mapsto ab^i$ $\square$

A polynomial $f \in \mathbb{Z}\langle x_1, \dots, x_n \rangle$ is monic if it has a term of highest degree with coefficient $= 1$.

A ring R is a PI-ring (polynomial identity ring) if there exists a monic $f \in \mathbb{Z}\langle x_1, \dots, x_n \rangle$ s.t.

$\forall a_1, \dots, a_n \in R: \quad f(a_1, \dots, a_n) = 0$

Commutative rings are PI ($x_1x_2 - x_2x_1$), subrings of PI rings are PI.
One can show: rings module-finite over commutative rings are PI, in particular $M_n(k)$ with k a commutative ring a PI (also subrings of $M_n(k)$)

Cor 2.10 If R is a PI-domain, then R is Ore

(Proof. 2.9(3))

The other POV:

Def 2.11 Let $R \subseteq Q$ be rings
R is a right order in Q if
(i) $R^\bullet \subseteq Q^\times$
(ii) $Q = \{ a s^{-1} : a \in Q, s \in R^\bullet \}$

R is an order in Q if it is a left & right order.

Prop 2.12 Let R be a ring.
Then R right Ore $\iff$ R is a right order in some ring Q.
In this case, $Q \cong Q_{cl}^r(R)$ over R.

<u>Examples:</u> (1) $R = \mathbb{Z}G$, G group, $S := \mathbb{Z}\setminus\{0\}$

$\Longrightarrow RS^{-1} \cong \mathbb{Q}G$

(2) $R = M_n(\mathbb{Z})$, $S = \mathbb{Z}\setminus\{0\} \Longrightarrow RS^{-1} \cong M_n(\mathbb{Q})$

$R = \mathbb{Z}1 \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}k = \mathbb{Z}[i,j,k]$ ← Lipschitz quaternions; $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $S = \mathbb{Z}\setminus\{0\}$

$\Longrightarrow RS^{-1} = \mathbb{Q}[i,j,k]$ quaternions

$$\mathbb{Z}[i,j,k] \subsetneq \mathbb{Z}\left[i,j,\frac{1+i+j+k}{2}\right] \subseteq \mathbb{Q}[i,j,k]$$

$\| \uparrow$ Hurwitz quaternions

$$\left\{\frac{a+bi+cj+dk}{2} : a \equiv b \equiv c \equiv d \mod 2\right\}$$

(2') <u>More generally:</u> K number field, $\mathcal{O}_K$ ring of integers,

$K \subseteq A$ CSA$/K$ $(Z(A) = K,\ [A:K] < \infty,\ A$ simple$)$

$(\to$ Artin-Wedderburn: $A = M_n(D)$, D div. ring, $Z(D) = K$, $[D:K] < \infty)$

An classical order R in A is a subring $\mathcal{O}_K \subseteq R \subseteq A$

s.t. (i) R is f.g. as $\mathcal{O}_K$-module

(ii) $RK = A = R(\mathcal{O}_K^\bullet)^{-1}$.

Classical orders are orders in A & one can show the converse.

(ms nc analogue of orders in number fields)

(3) $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$, $T := \{nI : n \in \mathbb{Z}\setminus\{0\}\} \Rightarrow RT^{-1} = \begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}$

$\Longrightarrow R$ is an order in Q, $Q = Q^r_{cl}(R) = Q^\ell_{cl}(R)$

$$S := \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a,b,c \in \mathbb{Z}, \boxed{a \neq 0}\right\}$$

$\varphi : R \to \mathbb{Z}$, $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \longmapsto a$ ring hom.,

$\ker\varphi = \operatorname{ass} S = \left\{\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} : b,c \in \mathbb{Z}\right\}$, $\varphi(S) = \mathbb{Z}\setminus\{0\}$

$\Longrightarrow RS^{-1} = \mathbb{Q}$

$S^{-1}R$ does not exist: Show S is not left reversible

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\in S} = 0$$

But $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq 0$ if $a \neq 0$

(4) Let D be a division ring, $\sigma \in End(D)$, $R := D[x;\sigma]$

$\Rightarrow R$ PLID $\Rightarrow R$ left Ore domain

(i) If $\sigma(D) = D$, then σ is an automorphism, and by symmetry R is also right Ore (in fact, PRID)

(ii) Suppose $\sigma(D) \subsetneq D$, $t \in D \setminus \sigma(D) \xRightarrow{L.16} \{x, tx\}$ are right linearly indep. $/R \Rightarrow R$ is not right Ore

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Skew polynomial rings (incl. derivation)

Let R be a domain, $\sigma \in End(R)$ and δ a σ-derivation on R. (i.e. $\forall_{a,b \in R}: \delta(a+b) = \delta(a) + \delta(b)$ and

$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ (twisted Leibniz rule))

Every element of $S = R[x;\sigma,\delta]$ can be written in the form $\sum_{i=0}^{n} a_i x^i$. Multiplication: $xa = \sigma(a)x + \delta(a)$

[Special cases: $\delta = 0$ or $\sigma = 1$, e.g. $\mathbb{C}(t)[x; \frac{d}{dt}]$]

If R is a domain, so is S.

Theorem 2.13 Let R be a left Ore domain, σ an injective endomorphism and δ a σ-derivation. Then $S = R[x;\sigma,\delta]$ is a left Ore domain.

If σ is an automorphism, the converse holds.

Proof: „$\Leftarrow$" Suppose σ is an automorphism

Let $a, b \in R^\circ$. Let $f, g \in S^\circ$ s.t. $fa = gb$.

From leading coefficients: $c\sigma^n(a) = d\sigma^n(b)$ for some $c, d \in R^\circ$, $n \geq 0$.

$$\Rightarrow \sigma^{-n}(c)\, a = \sigma^{-n}(d)\, b \rightarrow Ra \cap Rb \neq 0.$$

„$\Rightarrow$" Let R be left Ore, D the quotient div. ring.

- σ extends to D by $\underline{\sigma(as^{-1}) := \sigma(a)\sigma(s)^{-1}}$ $(a \in R, s \in R^\circ)$
- δ extends to a σ-derivation on D by

$$\underline{\delta(as^{-1}) := \delta(a)s^{-1} - \sigma(a)\sigma(s)^{-1}\delta(s)s^{-1}}$$

[How to find this?

δ σ-derivation $\Leftrightarrow \rho_\delta : \begin{cases} R \longrightarrow T_2(R) \\ a \longmapsto \begin{pmatrix} \sigma(a) & \delta(a) \\ 0 & a \end{pmatrix} \end{cases}$ is a ring hom.

ρ_δ extends to $D \to T_2(D)$ and we can compute

$$\begin{pmatrix} \sigma(a) & \delta(a) \\ 0 & a \end{pmatrix}\begin{pmatrix} \sigma(s) & \delta(s) \\ 0 & s \end{pmatrix}^{-1} = \begin{pmatrix} \sigma(a) & \delta(a) \\ 0 & a \end{pmatrix}\begin{pmatrix} \sigma(s)^{-1} & -\sigma(s)^{-1}\delta(s)s^{-1} \\ 0 & s^{-1} \end{pmatrix}$$]

So we can form $D[x; \sigma, \delta] \supseteq R[x; \sigma, \delta] = S$

$D[x; \sigma, \delta]$ is a PLID, hence left Ore, and has a left quotient ring Q.

Suffices to show: S is a left order in Q.

$S^\circ \subseteq Q^\times$ ✓

Let $q \in Q \Rightarrow q = f^{-1}g$ with $0 \neq f = \sum a_i x^i$, $g = \sum b_i x^i$, a_i, b_i $\boxed{\in D}$.

Choose $s \in R^\circ$ s.t. $sa_i = c_i \in R$, $sb_i = d_i \in R$ $\forall i$

(„common denominator")

$$\Rightarrow f^{-1}g = \left(s^{-1}\sum c_i x^i\right)^{-1}\left(s^{-1}\sum d_i x^i\right) = \left(\sum c_i x^i\right)^{-1}\left(\sum d_i x^i\right).$$

$\square$

Exm 2.14 Let k be a left Ore domain.

$\Rightarrow \frac{d}{dy}: k[y] \to k[y]$, $\sum_{i=0}^{n} a_i y^i \mapsto \sum_{i=1}^{n} i a_{i-1} y^{i-1}$ is a derivation.

$A_1(k) := k[y][x; \frac{d}{dy}]$ is the 1st Weyl algebra, and is an Ore domain

$\cong k\langle y, x\rangle / \langle xy - yx = 1\rangle$

The n-th Weyl algebra

$A_n(k) := k[y_1, \dots, y_n][x_1, \frac{d}{dy_1}] \dots [x_n, \frac{d}{dy_n}]$

$\cong A_{n-1}(A_1(k))$ is also an Ore domain.

Remark (1) If R is an Ore domain, $R[[x]]$ need not be Ore (Kerr '82)

(2) If R is an Ore ring with zero-divisors, $R[x]$ neither needs to be left nor right Ore! (Cedó & Herbera '95)

2.2.1 Extensions & Contractions

R ring, $S \subseteq R$ right denominator set, $Q := RS^{-1}$, $\varphi: R \to Q$

For $I \leq R_R$, $I^e := \varphi(I)Q$,

for $J \leq Q_Q$, $J^c := \varphi^{-1}(J)$

Proposition 2.15 $I, I_i \leq R_R$, $J, J_i \leq Q_Q$

(1) $J^{ce} = J$

(2) $I^e = \{a s^{-1} : a \in I, s \in S\}$

(3) $I^{ec} = \{r \in R : \exists s \in S : rs \in I\} \supseteq I$

(4) If $\bigoplus_i I_i \leq R_R$, then $\bigoplus_i I_i^e \leq Q_Q$

(5) If $\bigoplus_j J_j \leq Q_Q$, then $\bigoplus_j J_j^c \leq R_R$

(6) R right noetherian [artinian] $\Rightarrow$ Q right noetherian [artinian] (by (1))

(Proof skipped)

(!) If $I \trianglelefteq R$, then I^e need not be an ideal of Q

<u>Exm 2.16</u> $k := \mathbb{Q}[\{t_i : i \in \mathbb{Z}\}]$ polynomial ring, $\sigma(t_i) := t_{i+1}$,

$R := k[x; \sigma]$, $S := \{x^n : n \geq 0\}$

$\Rightarrow S$ right & left denominator set (Exc.), $RS^{-1} = k[x^{\pm 1}; \sigma]$

$I := \sum_{i \geq 1} t_i R \trianglelefteq R$ (since $x t_i = t_{i+1} x \in I$ for $i \geq 1$)

<u>I^e is not an ideal in RS^{-1}:</u>

$x^{-1} t_1 = \sigma^{-1}(t_1) x^{-1} = t_0 x^{-1} \notin I^e$

<u>Prop 2.16</u> (1) Assume Q is right noetherian.

If $I \trianglelefteq R$, then $I^e \trianglelefteq Q$

(2) If R is noetherian, then is a bijection

$$\begin{aligned} \operatorname{Spec} Q &\longleftrightarrow \{P \in \operatorname{Spec}(R) : P \cap S = \emptyset\} \\ J &\longmapsto J^c \\ I^e &\longmapsfrom I \end{aligned}$$

(Recall: $P \trianglelefteq R$ is prime if $\forall I, J \trianglelefteq R$: $IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$)

<u>Proof:</u> We only show (1)

Let $s \in S$. $sI \subseteq I$

$\Rightarrow I \subseteq s^{-1} I \subseteq s^{-2} I \subseteq \ldots \subseteq Q$

$\Rightarrow I^e \subseteq s^{-1} I^e \subseteq s^{-2} I^e \subseteq \ldots \subseteq Q$

Q right noetherian $\Rightarrow s^{-n} I^e = s^{-(n+1)} I^e$ for some $n \geq 0$

$\Rightarrow I^e = s^{-1} I^e \Rightarrow \forall r \in R$: $r s^{-1} I^e = r I^e \subseteq I^e$

$\Rightarrow Q I^e \subseteq I^e$.

(2 skipped) $\square$

3. Uniform Dimension

Let R be a ring

Def 3.1 Let N, M be R-modules.

$N \leq M$ is *essential* if $\forall\, 0 \neq K \leq M: \quad N \cap K \neq 0$

Notation: $\quad N \leq_e M$

Lemma 3.2 (Basic Properties) M R-module

(1) $N \leq_e M \wedge K \leq_e N \Rightarrow K \leq_e M$

(2) $N_1, N_2 \leq_e M \Rightarrow N_1 \cap N_2 \leq_e M$

(3) $N \leq_e M \Leftrightarrow \forall\, 0 \neq m \in M: \exists r \in R: 0 \neq mr \in N$

(4) $N \leq_e M,\ m \in M \Rightarrow m^{-1}N := \{r \in R : mr \in N\} \leq_e R_R$

(5) I set, $N_i \leq M_i$, $i \in I$:

$$(\forall i \in I : N_i \leq_e M_i) \Leftrightarrow \bigoplus_{i \in I} N_i \leq_e \bigoplus_{i \in I} M_i$$

(6) If $N \leq M$, $\exists N' \leq M$ s.t. $N \cap N' = 0$ & $N \oplus N' \leq_e M$

(7) M is the only essential submodule of M
$\Leftrightarrow M$ is semisimple

Proof: (1) Let $0 \neq X \leq M \Rightarrow X \cap N \neq 0 \Rightarrow K \cap (X \cap N) \neq 0$
$\Rightarrow K \cap X \neq 0 \Rightarrow K \leq_e M$

(2) Let $0 \neq X \leq M \Rightarrow Y := X \cap N_1 \neq 0 \Rightarrow Y \cap N_2 = X \cap N_1 \cap N_2 \neq 0$
$\Rightarrow N_1 \cap N_2 \leq_e M$

(3) „$\Rightarrow$" $N \cap mR \neq 0$ „$\Leftarrow$" $0 \neq X \leq M$, let $0 \neq m \in X \Rightarrow mR \cap N \neq 0$
$\Rightarrow X \cap N \neq 0$

(4) Let $0 \neq I \leq_r R_R$.

Case 1: $mI = 0 \Rightarrow I \subseteq m^{-1}N$ ✓

Case 2: $mI \neq 0 \overset{N \leq_e M}{\Longrightarrow} mI \cap N \neq 0 \Rightarrow I \cap m^{-1}N \neq 0$

(5) „$\Leftarrow$" ✓

„$\Rightarrow$" Sufficient to consider finite I (by (3)); reduce to $I = \{1,2\}$ by induction.

Suppose $0 \neq X \leq M_1 \oplus M_2$

Case 1: $X \cap M_i \neq 0$ for some $i \Rightarrow X \cap N_i \neq 0$ ✓

Case 2: $X \cap M_1 = 0 = X \cap M_2$. (*)

Let $0 \neq x = m_1 + m_2 \in X \Rightarrow m_1, m_2 \neq 0$

$\Rightarrow m_1 R \cap N_1 \neq 0 \Rightarrow \exists r \in R : 0 \neq m_1 r \in N_1$

$\overset{(*)}{\Rightarrow} m_2 r \neq 0 \Rightarrow \exists s \in S : 0 \neq m_2 rs \in N_2$

$\Rightarrow 0 \neq xrs = m_1 rs + m_2 rs \in X \cap (N_1 \cap N_2)$.

(6) $\Omega := \{ X \leq M : X \cap N = 0 \}$

Every chain in $(\Omega, \subseteq)$ has an upper bound, so Zorn's lemma implies $\exists$ maximal $N' \in \Omega$.

Suppose $(N \oplus N') \cap Y = 0 \Rightarrow (N' \oplus Y) \cap N = 0$

$\Rightarrow N' \oplus Y \in \Omega \Rightarrow Y = 0 \Rightarrow N \oplus N' \leq_e M$.

(7) M semisimple $\Leftrightarrow$ every submodule of M is a direct summand

So (6) implies the claim. $\square$

Remark (1) Let M be a module. If $M \leq E$ s.t. E is injective & $M \leq_e E$, then E is an injective hull of M. (Injective hulls are unique up to iso.)

E.g. $\mathbb{Q}_{\mathbb{Z}}$ is the injective hull of $\mathbb{Z}$

(2) Let R be a prime ring, $0 \neq I \trianglelefteq R$. Then $I_R \leq_e R_R$ and ${}_R I \leq_e {}_R R$

[Let $0 \neq J \leq R_R \Rightarrow 0 \neq JI \subseteq J \cap I$]

Def 3.3 A module U is uniform if $U \neq 0$ and $\forall 0 \neq V, W \leq U: \quad V \cap W \neq 0$

[$\Leftrightarrow$ every nonzero submodule of U is essential

$\Leftrightarrow \forall u, u' \in U \ \exists r, r' \in U: \quad ur = u'r' \neq 0$]

Theorem 3.4 ("Steinitz Repl Theorem") Let M be a R-module, $U = U_1 \oplus \dots \oplus U_m \leq_e M$, $V = V_1 \oplus \dots \oplus V_n \leq_e M$ with all U_i, V_j uniform.

Then $m = n$

Proof: W.r.t. $n \geq m$. $\hat{U} := U_2 \oplus \dots \oplus U_m$

Claim: $\exists j: \quad \hat{U} \cap V_j = 0$

Proof of Claim: Suppose $\forall j: \quad \hat{U} \cap V_j \neq 0$

$\Rightarrow \forall j: \quad \hat{U} \cap V_j \leq_e V_j \quad (V_j \text{ uniform})$

$$\Rightarrow \underbrace{(\hat{U} \cap V_1) \oplus \dots \oplus (\hat{U} \cap V_m)}_{\subseteq\, \hat{U} \cap V} \leq_e V_1 \oplus \dots \oplus V_m \leq_e M$$

$\Rightarrow \hat{U} \cap V \leq_e M \Rightarrow \hat{U} \leq_e M$ ↯ $\quad \square$ (Claim)

W.r.t. $V_1 \cap \hat{U} = 0$, Set $U' := \hat{U} \oplus V_1$

Then $U' \cap U_1 \neq 0$ (otherwise $U_1 + U' = U_1 + \dots + U_m + V_1$ is direct ↯ $U \leq_e M$)

$$\Rightarrow \underbrace{(U' \cap U_1) \oplus U_2 \oplus \dots \oplus U_m}_{\subseteq\, U'} \leq_e U_1 \oplus \dots \oplus U_m \leq_e M \Rightarrow U' \leq_e M$$

Repeating, inductively we get from U' to
$U'' = V_1 \oplus V_2 \oplus U_3 \oplus \ldots \oplus U_m \leq_e M$. (renumber!)

After m steps: $U^{(m)} = V_1 \oplus V_2 \oplus \ldots \oplus V_m \leq_e M$

Also $V_1 \oplus V_2 \oplus \ldots \oplus V_m \oplus \ldots \oplus V_n \leq_e M$

$\Longrightarrow m = n$. □

Def 3.4 M_R has <u>uniform dimension</u> (or Goldie rank) $n \in \mathbb{N}_0$, $\operatorname{udim} M = n$, if there exists $V \leq_e M$ s.t. $V = V_1 \oplus \ldots \oplus V_n$ with $V_1, \ldots, V_n$ uniform.
If no such n exists, $\operatorname{udim} M := \infty$.

Note: ·) $\operatorname{udim} M = 0 \Leftrightarrow M = 0$, $\operatorname{udim} M = 1 \Leftrightarrow M$ uniform
·) If R is a div. ring, $\operatorname{udim} M = \dim M$

Prop. 3.5 Let $\operatorname{udim} M = n < \infty$. If $N = N_1 \oplus \ldots \oplus N_k \leq M$ with $N_1, \ldots, N_k \neq 0$, then $k \leq n$.

Proof: By induction on n. $\underline{n=0}$ ✓

$\underline{n-1 \to n, n \geq 1}$: Let $V = V_1 \oplus \ldots \oplus V_n \leq_e M$, V_i uniform,

$N_i' := N_i \cap V$ $(1 \leq i \leq k)$

$\Rightarrow N' := N_1' \oplus \ldots \oplus N_k' \leq V_1 \oplus \ldots \oplus V_n$, $\hat{N} := N_2' \oplus \ldots \oplus N_k'$

as in the proof of Thm 3.3, $\exists j : \hat{N} \cap V_j = 0$,
Wr.l. $\hat{N} \cap V_1 = 0$

$\Longrightarrow \hat{N} \hookrightarrow V/V_1 \cong \underbrace{V_2 \oplus \ldots \oplus V_n}_{\operatorname{udim}\ n-1} \overset{IH}{\Longrightarrow} k-1 \leq n-1$. □

Prop 3.6 udim $M=\infty \iff M$ contains infinite direct sum of nonzero submodules

Proof: „$\Leftarrow$" ✓ „$\Rightarrow$" By contradiction. Suppose M does not contain an inf. direct sum.

Claim: If $0 \neq N \leq M$, then N contains a uniform submodule

Proof: Suppose not. Then N is not uniform, so $\exists A_1, B_1 \leq N$,
$A_1, B_1 \neq 0$, $A_1 \oplus B_1 \leq N$
B_1 not uniform $\Rightarrow \exists\, 0 \neq A_2, B_2$: $B_1 \geq A_2 \oplus B_2$
Inductively: $A_1 \oplus \ldots \oplus A_m \oplus B_m \leq M$ $(A_i, B_m \neq 0)$
↯ $\mathrm{udim}(M) < \infty$

$\square$ (Claim)

Let $0 \neq V_1 \leq M$ be uniform. If $V_1 \not\leq_e M$, then $M \supseteq V_1 \oplus V_2'$, $V_2' \neq 0$ and we may choose $V_2 \subseteq V_2'$ uniform, so $M \supseteq V_1 \oplus V_2$.
If $V_1 \oplus V_2 \not\leq_e M$,
We can continue, to find $V_1 \oplus \ldots \oplus V_n \leq M$, V_i uniform, until $V_1 \oplus \ldots \oplus V_n \leq_e M$. This must happen for some n by our assumption. So udim $M = n$ ↯

$\square$

Cor 3.7 $\mathrm{udim}\, M = \sup\{k \in \mathbb{N}_0 : M$ contains a direct sum of k nonzero submodules$\} \in \mathbb{N}_0 \cup \{\infty\}$

Proof: Let $k_0 =$ r.h.s.
Case $k_0 = \infty$: $(3.5) \Rightarrow \mathrm{udim}\, M = \infty$
Case $k_0 < \infty$: $(3.7) \Rightarrow \mathrm{udim}\, M < \infty \overset{(3.5)}{\Longrightarrow} \mathrm{udim}\, M = k_0$ $\square$

Cor 3.8 (1) If M_R is noetherian or artinian, then udim $M < \infty$

(2) If M has finite composition length $n < \infty$, then udim $M \leq n$.
Further, udim $M = n \iff M$ semisimple

Proof: (1) ✓

(2) Suppose $N_1 \oplus \ldots \oplus N_k \leq M$, $N_i \neq 0$

$$\Rightarrow \quad k \leq \sum_{i=1}^{k} \text{length}(N_i) \leq \text{length}(M) = n. \qquad (*)$$

$\overset{(3.7)}{\Longrightarrow}$ udim $M \leq n$.

Suppose M is semisimple, i.e., a direct sum of n simple modules.
$\Rightarrow$ udim $M \geq n$, so udim $M = n$.

Conversely, suppose $n = $ udim M.
$\Rightarrow \exists N_i \neq 0 : N_1 \oplus \ldots \oplus N_n \leq M$.
$\overset{(*)}{\Longrightarrow} \forall i$: length$(N_i) = 1$, i.e. N_i simple and

length$\left(M/_{N_1 \oplus \ldots \oplus N_n}\right) = 0$, so $M = N_1 \oplus \ldots \oplus N_n$.

Thus M is semisimple. $\square$

Examples (1) $R = \mathbb{Z}$, $p \in \mathbb{P}$: length$\left(\mathbb{Z}/_{p^r\mathbb{Z}}\right) = r$,
udim$\left(\mathbb{Z}/_{p^r\mathbb{Z}}\right) = 1$

Suppose $\overset{\mathbb{N}}{\overset{\in}{m}} = p_1^{e_1} \cdots p_r^{e_r}$, p_i pw. distinct primes, $e_i \geq 1$.
Then length$\left(\mathbb{Z}/_{m\mathbb{Z}}\right) = e_1 + \ldots + e_r$ but udim$\left(\mathbb{Z}/_{m\mathbb{Z}}\right) = r$.

(2) M_R f.g. $\not\Rightarrow$ udim $M < \infty$:

$R = \prod_{i \in \mathbb{N}_0} \mathbb{Z}$ $\qquad \bigoplus_{i \in \mathbb{N}_0} \mathbb{Z} \trianglelefteq R$

$\Rightarrow$ udim$(R_R) = \infty = $ udim$({}_RR)$, but R_R, ${}_RR$ are cyclic

Cor 3.9 (1) $\operatorname{udim}(M_1 \oplus \dots \oplus M_k) = \operatorname{udim} M_1 + \dots + \operatorname{udim} M_k$

(2) Let $N \leq M$. Then

(i) $\operatorname{udim} N \leq \operatorname{udim} M$, and $\operatorname{udim} N = \operatorname{udim} M \Leftrightarrow N \leq_e M$

(ii) If $N \not\leq_e M$, then $\operatorname{udim} N < \operatorname{udim} M$ or $\operatorname{udim} N = \operatorname{udim} M = \infty$

Proof: (1) If all $\operatorname{udim} M_i < \infty$, this follows from the definition.
If $\exists i$: $\operatorname{udim} M_i = \infty$, this follows from Prop. 3.6.

(2) (i) From Cor. 3.7.

(ii) Suppose $N \leq M$, but $N \not\leq_e M$

- If $\operatorname{udim} N = \infty$, then $\operatorname{udim} M = \infty$
- Let $\operatorname{udim} N = n < \infty$, $N_1 \oplus \dots \oplus N_n \leq N$, $N_i \neq 0$

$N \not\leq_e M \Rightarrow \exists N' \neq 0$, $N' \leq M$: $N \cap N' = 0$

$\Rightarrow M \geq N' \oplus N_1 \oplus \dots \oplus N_n \Rightarrow \operatorname{udim}(M) \geq n+1$.

Examples: (1) udim is not additive on SES:

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0,$$

but all modules have udim 1.

(2) $M \twoheadrightarrow N \not\Rightarrow \operatorname{udim} N \leq \operatorname{udim} M$:

$R = M = \mathbb{Z}$, $N = \mathbb{Z}/p_1 \cdots p_r\mathbb{Z}$: $\operatorname{udim}(M) = 1$, $\operatorname{udim}(N) = r$.

Similar: $\mathbb{Q}_{\mathbb{Z}} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$, $\operatorname{udim} \mathbb{Q}_{\mathbb{Z}} = 1$

$$\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p^\infty) \Rightarrow \operatorname{udim} (\mathbb{Q}/\mathbb{Z})_{\mathbb{Z}} = \infty$$

Example: Ring with different (finite) left & right uniform dimension

$$S := \begin{pmatrix} \mathbb{C} & {}_{\mathbb{C}}\mathbb{C}_{\mathbb{R}} \\ 0 & \mathbb{R} \end{pmatrix}$$

left: $${}_S S = \underbrace{\begin{pmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{pmatrix}}_{\text{simple}} \oplus \underbrace{\begin{pmatrix} 0 & \mathbb{C} \\ 0 & \mathbb{R} \end{pmatrix}}_{=: M}$$

$\begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix} \leq_e M$: Let $\alpha = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \in M \setminus \{0\}$

Case 1: $x \neq 0$: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \alpha = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix} \setminus \{0\}$

Case 2: $x = 0$: $\Rightarrow y \neq 0 \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \alpha = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix} \setminus \{0\}$

${}_S\begin{pmatrix} 0 & {}_{\mathbb{C}}\mathbb{C}_{\mathbb{R}} \\ 0 & 0 \end{pmatrix}$ is simple, bec ${}_{\mathbb{C}}\mathbb{C}$ is simple

$\Longrightarrow \operatorname{udim}_S S = 2$

right: $$S_S = \underbrace{\begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & 0 \end{pmatrix}}_{=: N} \oplus \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{R} \end{pmatrix}}_{\text{simple}}$$

Now $\begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix} \leq_e N$ (as above), but

$$\operatorname{udim} \begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix}_S = \dim \mathbb{C}_{\mathbb{R}} = 2$$

$\Longrightarrow \operatorname{udim} S_S = 3$

More general: $K \subseteq L$ fields, $[L : K] = n$, $S = \begin{pmatrix} L & {}_L L_K \\ 0 & K \end{pmatrix}$

$\Rightarrow \operatorname{udim}_S S = 2$, $\operatorname{udim} S_S = n+1$

Prop 3.10 $\operatorname{udim}(M_R) = n < \infty \Leftrightarrow E(M)$ is direct sum of n indecomposable injective modules

We will use:
- •) An injective module I is indecomposable $\Leftrightarrow I$ is uniform [L.M.R, (3.52)]
- •) $E(M \oplus N) = E(M) \oplus E(N)$ (by Lemma 3.2 (5) or [LMR, (3.4)])
- •) If $M \leq_e N$, then $E(M) = E(N)$ [L.M.R. (3.33)]

Proof: $\operatorname{udim} M_R = \operatorname{udim} E(M)$ by Cor 3.9(2), so w.l.o.g. M is injective.

„$\Leftarrow$" Let $M = I_1 \oplus \ldots \oplus I_n$ with I_i indecomposable injectives
$\Rightarrow$ each I_i is uniform $\Rightarrow \operatorname{udim} M = \sum_{i=1}^{n} \operatorname{udim} I_i = n$.

„$\Rightarrow$" Suppose $\operatorname{udim} M = n$.
Let $V_1 \oplus \ldots \oplus V_n \leq_e M$, V_i uniform.
$\Rightarrow E(M) = E(V_1 \oplus \ldots \oplus V_n) = E(V_1) \oplus \ldots \oplus E(V_n)$
But since $\operatorname{udim} E(V_i) = \operatorname{udim} V_i = 1$, each $E(V_i)$ is uniform, hence indecomposable $\square$

Prop 3.11 Let R be a ring with right denominator set S, $Q = RS^{-1}$. If M_R is a module s.t. $\forall m \in M \, \forall r \in S$ $mr = 0 \Rightarrow m = 0$ („S-torsion-free"), then $\operatorname{udim}(M_R \otimes_R Q) = \operatorname{udim}(M_R)$.

Proof: The S-torsion-free property allows us to embed $M \hookrightarrow M \otimes_R Q =: M^Q$ [M.R., Prop 2.1.17 – construct MS^{-1} and then $MS^{-1} \cong M \otimes_R Q$ by Universal Property]

Case 1: $\operatorname{udim} M_R = \infty \Rightarrow M$ contains $V_1 \oplus V_2 \oplus V_3 \oplus \ldots$, $V_i \neq 0$
$\Rightarrow M^Q \geq V_1^Q \oplus V_2^Q \oplus \ldots \Rightarrow \operatorname{udim}(M^Q_Q) = \infty$

Case 2: $\operatorname{udim} M = n < \infty$.

Let $V_1 \oplus \dots \oplus V_n \leq M$, $V_i \neq 0$, $\rightarrow$ $V_1^Q \oplus \dots \oplus V_n^Q \leq M^Q$

$\Rightarrow \operatorname{udim}(M^Q) \geq n$.

Suppose $U_1 \oplus \dots \oplus U_k \leq M^Q$, $U_i \neq 0$.

Then $(U_i \cap M) \leq M$ and $U_i \cap M \neq 0$ by clearing denominators

$\Rightarrow (U_1 \cap M) \oplus \dots \oplus (U_k \cap M) \leq M$, $U_i \cap M \neq 0$

$\Rightarrow n = \operatorname{udim} M \geq k$

So $\operatorname{udim}(M^Q) = n$. $\square$

Prop 3.12 Let $S \subseteq R$ be a right denominator set $\boxed{\text{s.t. } S \subseteq R^\cdot}$

$Q := RS^{-1} \supseteq R$, $I \subseteq I' \leq R_R$, $J \subseteq J' \leq Q_Q$, $I^e := IQ$, $J \cap Q =: J^c$.

Then

(1) $I \leq_e I' \Leftrightarrow I^e \leq_e I'^e$, $J \leq_e J' \Leftrightarrow J^c \leq J'^c$ ...

(2) $\operatorname{udim} I_R = \operatorname{udim} I^e_Q = \operatorname{udim} I^e_R$,

$\operatorname{udim} J_Q = \operatorname{udim} J_R = \operatorname{udim} J^c_R$

Proof skipped (LMR, (10.34), (10.35)). Idea: Clear denominators!

Example 3.13 $S \subseteq R^\cdot$ is necessary.

Let $R = \mathbb{Z}\langle x,y\rangle / \langle yx, y^2\rangle = \{ f(x) + g(x)y : g, f \in \mathbb{Z}[x] \}$

(as in Exm. 2.6), $S = \{x^n : n \geq 0\}$ right denominator set

$I := Ry = \bigoplus_{i \geq 0} x^i y \mathbb{Z}$ $\Rightarrow \operatorname{udim}(I_R) = \operatorname{udim}(R_R) = \infty$

$\uparrow$ right ideals of R

But $I^e = 0$

$I \leq_e R_R$, $I^e \not\leq_e Q_Q$

$\operatorname{udim} Q_Q = 1 = \operatorname{udim} \mathbb{Z}[x^{\pm 1}]$

3.1 Complement submodules

(Let R be a ring.)

Def 3.13 Let M_R be an R-module, $N \leq M$.

- $C \leq M$ is a complement to N (in M) if C is maximal with the property $C \cap N = 0$
- $C \leq M$ is a complement in M ($C \leq_c M$) if there exists some $N \leq M$ s.t. C is a complement to N in M.

By Zorn's lemma, every module has a complement. | If $N' \leq M$, $C' \cap N = 0 \to$ enlarge C' into complement C

Exm:
- If $M = N \oplus C$, C is a complement to N
- If $N \leq_e M$, then 0 is the only complement of N in M
- $M_{\mathbb{Z}} = \mathbb{Z} \oplus \mathbb{Z}$: $C = 0 \oplus \mathbb{Z}$ is a complement to $n\mathbb{Z} \oplus 0$ for all $n \neq 0$.

  $0 \oplus n\mathbb{Z}$ with $|n| > 1$ is not a complement

  [If $(0 \oplus n\mathbb{Z}) \cap M = 0 \Rightarrow (0 \oplus \mathbb{Z}) \cap M = 0$]

Lemma 3.14 (1) Let $C \leq N \leq M$. Then $C \leq_c M \to C \leq_c N$

(2) If $C \leq_c N$ and $N \leq_c M$, then $C \leq_c M$

Proof. (1) Let $X \leq M$ s.t. C is a complement to X in M.

Then C is a complement to $X \cap N$ in N.

[$C \cap (X \cap N) = 0$, and if $N \supseteq C' \supsetneq C$ with $C' \cap (X \cap N) = 0 \to C' \cap X = 0$]

(2) Let C be a complement to S in N, and N a complement to T in M. Claim: C is a complement to $S \oplus T$ in M.

Let $D \leq M$ with $C \subsetneq D$. To show: $D \cap (S \oplus T) \neq 0$

Wlog. $D \cap N = C$ (otherwise $(D \cap N) \cap S \neq 0$)

Let $d \in D \setminus N$.

$\Rightarrow (N + dR) \cap T \neq 0$

$\Rightarrow \exists n \in N, r \in R, t \in T: \quad n + dr = t \neq 0 \quad (*)$

If $n \in C$, then $0 \neq n + dr \in D \cap T$, so we are done.

Suppose $n \notin C \Rightarrow (C + nR) \cap S \neq 0$

$\Rightarrow \exists c \in C, r' \in R, s \in S: \quad c + nr' = s \neq 0. \quad (**)$

$$(**) - (*)r': \quad \underset{\underset{D}{\cap}}{c - drr'} = \underset{\underset{\neq 0}{\underset{S \oplus T,}{\cap}}}{s - tr'} \in (D \cap (S \oplus T)) \setminus \underline{0}$$

$\square$

<u>Prop 3.15</u> Assume $\operatorname{udim} M = n < \infty$. Then any chain of complements in M has length $\leq n$. I.e.

If $C_0 \lneq C_1 \lneq \ldots \lneq C_k$, $C_i \leq_c M$, then $k \leq n$.

<u>Proof:</u> Lemma 3.14(1) $\Rightarrow C_{i-1} \leq_c C_i$.

Let $S_i \leq C_i$ s.t. C_{i-1} is a complement to S_i $\boxed{\text{in } C_i}$.

If $S_i = 0$, then $C_{i-1} = C_i$ ↯, so $S_i \neq 0$ $(\forall i)$

$\Rightarrow S_1 \oplus \ldots \oplus S_k \leq M$

(For directness, note $S_1 \oplus \ldots \oplus S_{i-1} \leq C_{i-1}$, and $C_{i-1} \cap S_i = 0$),

so $k \leq n$ (Prop. 3.5).

Theorem 3.16 For M_R, TFAE:

(a) udim $M = \infty$

(b) $\exists$ infinite strictly ascending chain of complements in M

(c) — " ——— decending ——— . ———

Proof: (a) $\Rightarrow$ (b) Let $U_1 \oplus U_2 \oplus \ldots \leq M$ an infinite direct sum ($U_i \neq 0$; Prop 3.6).

Take $U_1 \leq C_1 \leq M$ a complement of $U_2 \oplus U_3 \oplus \ldots$ in M.

$\Rightarrow$ $C_1 \oplus \underset{\neq 0}{U_2} \oplus U_3 \oplus \ldots \leq M$ is direct

Pick $C_1 \oplus U_2 \leq C_2$ a complement of $U_3 \oplus U_4 \oplus \ldots$ in M,

Inductively: $C_{i-1} \lneq C_i$, $C_{i-1} \oplus U_i \leq C_i$ and C_i is a complement of $U_{i+1} \oplus \ldots$

(b) $\Rightarrow$ (c) Let $C_0 \lneq C_1 \lneq \ldots$ with each $C_i \leq_c M$. Then $C_{i-1} \leq_c C_i$

Let $S_1 \leq C_1$ s.t. C_0 is a complement to S_1 in C_1 ($C_0 \oplus S_1 \leq C_1$)

Inductively, construct S_i ($i \geq 1$) so that $\underbrace{C_0 \oplus S_1 \oplus \ldots \oplus S_{i-1}}_{\in C_{i-1}} \oplus S_i \leq C_i$ and C_{i-1} is a complement to S_i in C_i.

$\Rightarrow S := S_1 \oplus S_2 \oplus \ldots \leq M$, $C_0 \cap S = 0$

Take $S \leq Y_1 \leq M$, with Y_1 a complement to C_0 in M.

In Y_1, enlarge $S_2 \oplus S_3 \oplus \ldots$ into a complement Y_2 of S_1.

Inductively: $S_i \oplus S_{i+1} \oplus \ldots \leq Y_i \leq Y_{i-1}$ is a complement of S_{i-1} in Y_{i-1}.

Then: $Y_i \leq_c Y_{i-1}, Y_1 \leq_c M \xRightarrow{\text{(Transitivity)}} Y_i \leq_c M$.

$S_{i-1} \cap Y_i = 0$, $S_{i-1} \subseteq Y_{i-1} \Rightarrow Y_i \subsetneq Y_{i-1}$.

(c) $\Rightarrow$ (a) Prop 3.15.

$\square$

Corollary 3.17 M_R R-module

(1) TFAE:

(a) $\operatorname{udim} M < \infty$

(b) M satisfies ACC on complement submodules

(c) — " — DCC — " —

(2) $\operatorname{udim} M = \sup\{k \in \mathbb{N}_0 : M$ contains a chain of complement submodules of length $k\}$

Proof: (1) Negation of 3.16

(2) As in 3.16, $\square$

4. Singular Submodules, Nonsingular Rings

Let R be a ring.

Def 4.1 Let M be a R-module, $m \in M$.

- $\operatorname{ann}(m) := \{r \in R : mr = 0\}$
- m is singular if $\operatorname{ann}(m) \leq_e R_R$
- $Z(M) := \{m \in M : m \text{ is singular}\}$ is the singular submodule of M
- $Z(R) := Z(R_R)$ is the right singular ideal of R

Lemma 4.2 (1) $Z(M) \leq M$ is a submodule

(2) If $f \in \operatorname{Hom}_R(M,N)$, then $f(Z(M)) \subseteq Z(N)$

(3) If $M \leq N$, then $Z(M) = M \cap Z(N)$

(4) $Z(R_R)$ is an ideal of R.

(5) If $R \neq 0$, then $Z(R) \neq R$.

Proof: (1) ·) Let $m_1, m_2 \in \mathcal{Z}(M) \Rightarrow \operatorname{ann}(m_i) \leq_e R_R$

$\overset{L3.2(2)}{\Longrightarrow} \operatorname{ann}(m_1) \cap \operatorname{ann}(m_2) \leq_e R_R$

· $\operatorname{ann}(m_1+m_2) \supseteq \operatorname{ann}(m_1) \cap \operatorname{ann}(m_2) \Rightarrow m_1+m_2 \in \mathcal{Z}(M)$

·) Let $m \in \mathcal{Z}(M)$, $r \in R$. Show: $mr \in \mathcal{Z}(M)$

Let $s \in R \setminus \operatorname{ann}(mr)$. Show: $\exists t \in R$: $0 \neq st \in \operatorname{ann}(mr)$

$0 \neq (mr)s = m(rs) \xRightarrow{\operatorname{ann}(m) \leq_e R_R} \exists t \in R: \quad 0 \neq rst \ \wedge \ 0 = mrst$

$\Rightarrow 0 \neq st \ \wedge \ st \in \operatorname{ann}(mr)$

(2) $\operatorname{ann}(m) \subseteq \operatorname{ann}(f(m)) \quad \forall m \in M$

(3) ✓

(4) $\mathcal{Z}(R_R)$ is a right ideal by 1), so show: $\forall m \in \mathcal{Z}(M), r \in R: rm \in \mathcal{Z}(M)$.

But $\operatorname{ann}(rm) \supseteq \operatorname{ann}(m)$.

(5) $r\operatorname{ann}(1) = 0$ is not essential unless $R = 0$.

Def 4.3 ·) M_R is singular if $\mathcal{Z}(M) = M$ („nonsingular" ≠ „not singular")

·) M_R is non-singular if $\mathcal{Z}(M) = 0$

·) R is right non-singular if $\mathcal{Z}(R_R) = 0$

[$\Leftrightarrow$ every $I \leq_e R_R$ has $l\operatorname{ann}(I) = 0$]

·) R is nonsingular if it is left & right nonsingular.

Example 4.4 (1) Simple rings are nonsingular

(2) Let R be a commutative domain. Then every $0 \neq I \trianglelefteq R$ is essential. If M is an R-module,

$\mathcal{Z}(M) = \{m \in M : \operatorname{ann}(m) \neq 0\}$

is the torsion submodule of M.

So M singular $\Leftrightarrow$ M torsion

M nonsingular $\Leftrightarrow$ M torsion-free

Lemma 4.5 M_R singular $\Leftrightarrow$ $\exists A, B$ R-modules w $A \leq_e B$ s.t. $M \cong B/A$

Proof: "$\Leftarrow$" Wlog. $M = B/A$. Let $b \in B \setminus A$.

Show: $\mathrm{ann}(b+A) = \{r \in R : br \in A\} \leq_e R_R$.

Let $r \in R \setminus \mathrm{ann}(b+A) \Rightarrow br \notin A \overset{A \leq_e B}{\Longrightarrow} brs \in A \setminus \{0\}$ for some $s \in R$

$\Rightarrow 0 \neq rs \in \mathrm{ann}(b+A) \Rightarrow rR \cap \mathrm{ann}(b+A) \neq 0$

"$\Rightarrow$" Fix a presentation $0 \to K \to F \overset{\pi}{\to} M \to 0$ with F free.

Let $B \subseteq F$ be a basis.

For $x \in F \setminus K$, we show: $xR \cap K \neq 0$, then $K \leq_e F$.

Let $x = b_1 r_1 + \dots + b_n r_n$ with $b_i \in B$ pairwise distinct, $r_i \in R \setminus \{0\}$.

Call n the length of x. We may assume that n is the minimal length of an element $x \in F \setminus K$ (noether induction).

$\mathrm{ann}(\pi(b_n r_n)) = \{s \in S : b_n r_n s \in K\} \leq_e R_R$

So $\mathrm{ann}(\pi(b_n r_n)) \cap r_n R \neq 0$

$\Rightarrow \exists s_n \in S: \ r_n s_n \neq 0$ and $b_n r_n s_n \in K$

Since $b_n r_n s_n \neq 0$, also $x s_n \neq 0$, and

$$x s_n = b_1 r_1 s_1 + \dots + b_{n-1} r_{n-1} s_{n-1}$$

has length $n-1$.

$\overset{IH}{\Longrightarrow} x s_n R \cap K \neq 0$. $\square$

Prop. 4.6 R semisimple $\Leftrightarrow$ every right module M_R is nonsingular

Proof. "$\Rightarrow$" Let $m \in Z(M) \Rightarrow \mathrm{ann}(m) \leq_e R_R$ & $\mathrm{ann}(m)$ is a direct summand of R_R

$\Rightarrow \mathrm{ann}(m) = R_R \Rightarrow m = 0$

"$\Leftarrow$" Let $I \leq R_R$, and let J be a complement of I in R

$\Rightarrow I \oplus J \leq_e R_R \overset{L4.5}{\Longrightarrow} R_R / I \oplus J$ singular $\Longrightarrow R_R / I \oplus J = 0$

$\Rightarrow I \oplus J = R_R$, so R is semisimple $\square$

Prop 4.7 (1) If R is reduced (no nonzero nilpotent elements), it is non-singular

(2) If a central $x \in R$ is nilpotent, then $x \in Z(R_R)$

In particular: A commutative ring is reduced if and only if it is non singular

Proof (1) Show: $\forall x \in R$: $\mathrm{rann}(x) \cap xR = 0$ [then $\mathrm{rann}(x)$ is not essential in R for $x \neq 0$]

Let $y \in \mathrm{rann}(x) \cap xR$, $y = xz$ $(z \in R)$

$\Rightarrow (yx)^2 = y\underbrace{xy}_{=0}x = 0 \overset{R \text{ reduced}}{\Longrightarrow} yx = 0 \Rightarrow y^2 = y.xz = 0 \overset{\text{reduced}}{\Longrightarrow} y = 0$

(2) Show: $\mathrm{rann}(x) \leq_e R_R$.

Let $y \in R\setminus\{0\}$, and n minimal with $x^{n+1}y = 0$

$\Rightarrow 0 \neq x^n y \in \mathrm{rann}(x)$

$x^n y = y x^n \Rightarrow x^n y \in yR$ □

A right annihilator is a set of the form $\mathrm{rann}(X)$, $X \subseteq R$.
$(\mathrm{rann}(X) = \{r \in R : \forall x \in X : xr = 0\})$

$Y \subseteq R$ is right annihilator $\Leftrightarrow Y = \mathrm{rann}(\mathrm{lann}(Y))$

So: R satisfies ACC on right annihilators
$\Leftrightarrow$ — " — DCC " left — " —

Recall: R semiprime $\Leftrightarrow \forall I \triangleleft R$: $I^2 = 0 \Rightarrow I = 0$

Theorem 4.8 If R satisfies the ACC on right annihilators, then $Z(R_R)$ is a nilpotent ideal.

Proof: $I := Z(R_R)$.

$\Rightarrow \mathrm{rann}(I) \subseteq \mathrm{rann}(I^2) \subseteq \mathrm{rann}(I^3) \subseteq \ldots$ stabilizes,
i.e. $\exists m_0 \geq 1\ \forall m \geq m_0$: $\mathrm{rann}(I^m) = \mathrm{rann}(I^{m+1})$

Claim: $I^m = 0$:

Assume $I^m \neq 0$. Consider

$$\Omega = \{ \mathrm{rann}(z) : z \notin \mathrm{rann}_R(I^m) \},$$

and choose a maximal $\mathrm{rann}(x) \in \Omega$ (with $I^m x \neq 0$)

Let $a \in I \subseteq Z(R_R) \Rightarrow \mathrm{rann}(a) \leq_e R_R$, so
$\mathrm{rann}(a) \cap xR \neq 0$

$\Rightarrow \exists y \in R: \quad axy = 0, \quad xy \neq 0$

$\Rightarrow \mathrm{rann}(x) \subsetneq \mathrm{rann}(ax) \overset{\text{maximality}}{\Longrightarrow} I^m a x = 0.$

This holds for all $a \in I$, so $x \in \mathrm{rann}(I^{m+1}) = \mathrm{rann}(I^m)$ $\lightning$. $\square$

<u>Corollary 4.9</u> A semiprime ring satisfying the ACC on right annihilators is right nonsingular.

<u>Remark:</u> Cor 4.9 also holds if R is semiprime and satisfies the ACC on right annihilators of elements (sets of the form $\mathrm{rann}(x)$, $x \in R$) ([L.M.R, (7.19)], needs that the lower nilradical of a semiprime ring is zero)

5. Goldie's Theorem

<u>Recall:</u> .) If $R \subseteq Q$ are rings, R is a <u>right order</u> if $R^\circ \subseteq Q^\times$ and $Q = \{ as^{-1} : a \in R, s \in R^\circ \}$

.) $R \subseteq Q$ is a right order $\Leftrightarrow$ R right Ore.

<u>Then:</u> $Q = Q^r_{cl}(R)$

When does a ring Q have a right order?

Prop 5.1 For a ring Q TFAE:

(a) Q has a right order R

(b) — " — left — " —

(c) $Q^\circ = Q^\times$

(d) $Q = Q^{\ell}_{ce}(Q)$

(e) $Q = Q^{r}_{ce}(Q)$

If Q satisfies these conditions, Q is a ring of quotients (classical ring)

Proof. Suff to show (a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (e)

(c) $\Rightarrow$ (e) $\Rightarrow$ (a) ✓

(a) $\Rightarrow$ (c) $Q = Q^r_{ce}(R)$. Let $q \in Q^\circ$, $q = as^{-1}$ $(s \in R^\circ, a \in R)$

$\Rightarrow$ $s \in Q^\times \subseteq Q^\circ$ $\Rightarrow$ $a = (as^{-1}) \cdot s \in Q^\circ \cap R \subseteq R^\circ$

$\Rightarrow$ $q' = sa^{-1} \in Q$ and $qq' = 1 = \underset{=q}{q'q}$ $\Rightarrow$ $q \in Q^\times$. $\square$

Exm 5.2 •) Right Artinian rings are right quotient rings:

Let Q be artinian, $q \in Q^\circ$

$\Rightarrow$ $qQ \supseteq q^2Q \supseteq \dots$ stabilizes

$\Rightarrow$ $\exists m \geq 0$: $q^m Q = q^{m+1} Q$ $\Rightarrow$ $q^m = q^{m+1}x$, $x \in Q$

$\overset{}{\underset{q \in Q^\circ}{\Longrightarrow}}$ $1 = qx$.

•) A ring Q is semisimple iff it is artinian and $J(Q) = 0$ (Jacobson radical)

Which rings have semisimple rings of quotients?

Recall

Thm 5.3 (1) A ring Q is <u>semisimple</u> if it satisfies the following equivalent conditions:

(a) Q_Q is a sum of simple modules
(b) — " — direct sum — " ————
(c) every right Q-module is projective
(d) ——— " ——————— injective
(e) ——————————— semisimple
(f) every right ideal of Q is a direct summand
(g) Q is artinian & $J(Q)=0$

[By (g): symmetric condition on left]

(h) $Q \cong M_{n_1}(D_1) \times \dots \times M_{n_k}(D_k)$, $n_1, \dots, n_k \geq 1$, $D_1, \dots, D_n$ div. rings.

(2) Q is simple Artinian $\Leftrightarrow$ $Q \cong M_n(D)$, D division ring.

A semisimple ring has finitely many isomorphism classes of simple module (corresponding to row spaces $D_i^{1\times n_i}$ in (h)), and every module is uniquely a direct sum of such simple modules.

Def 5.4 A ring R is <u>right Goldie</u> if $\operatorname{udim} R_R < \infty$ <u>and</u> R satisfies the ACC on right annihilator ideals.

Thm 5.5 (Goldie's Theorem) For a ring R, TFAE:

(a) R is a right order in a semisimple ring
(b) R is semiprime right Goldie
(c) R is semiprime, right nonsingular, and $\operatorname{udim} R_R < \infty$.
(d) [Goldie's Lemma] For every $I \leq R_R$: $I \leq_e R_R \Leftrightarrow I \cap R^\bullet \neq \emptyset$

Remark: With a bit more work, one can also show that the conditions are equivalent to R being semiprime, $\operatorname{udim} R_R < \infty$, and R satisfies the ACC on right annihilators of elements (cf. Remark after Cor. 4.9)

To prove the crucial (c) ⇒ (d), we first show:

Prop 5.6 Let R be a semiprime, right nonsingular ring with $\operatorname{udim} R_R < \infty$.
(1) Let $a \in R$ be right regular (i.e. $\operatorname{rann}(a) = 0$). Then $aR \leq_e R_R$ and $a \in R^\circ$.
(2) Every $I \leq R_R$ contains $a \in I$ s.t. $\operatorname{rann}(a) \cap I = 0$
(3) [Goldie's Lemma] For every $I \leq R_R$: $I \leq_e R_R \Leftrightarrow I \cap R^\circ \neq \emptyset$

Proof: (1) $\operatorname{rann}(a) = 0 \Rightarrow aR_R \cong R_R \Rightarrow \operatorname{udim}(aR_R) = \operatorname{udim}(R_R) \Rightarrow aR_R \leq_e R_R$
Suppose $r \in \operatorname{lann}(a) \Rightarrow r(aR) = 0 \xrightarrow{aR \leq_e R_R} r \in Z(R_R) = 0 \Rightarrow r = 0$.

(2) Induction on $\operatorname{udim} I_R$. $n := \operatorname{udim} I_R$.
$n=0$: $\Rightarrow I = 0$ ✓
$n=1$: I_R is uniform, $I^2 \neq 0$ (since R is semiprime).
$\Rightarrow \exists a, a' \in I: \; aa' \neq 0$.
Claim: $\operatorname{rann}(a) \cap I = 0$
Proof of Claim: Let $J := \operatorname{rann}(a) \cap I$ and suppose $J \neq 0$.
$\Rightarrow J \leq_e I$ (I uniform)
$\Rightarrow (a')^{-1} J = \{ r \in R : a'r \in J \} \leq_e R_R$ (Lemma 3.2 (4))
$aa'((a')^{-1}J) \subseteq aJ = 0 \Rightarrow aa' \in Z(R_R) = 0$ ↯ □(Claim)
$n > 1$; $n-1 \to n$: Fix $I' \leq I$ with $\operatorname{udim}(I') = n-1$, and let $a' \in I'$ with $\operatorname{rann}(a') \cap I' = 0$.
If $\operatorname{rann}(a') \cap I = 0$, we are done. Suppose $\operatorname{rann}(a') \cap I \neq 0$

$U := \mathrm{rann}(a') \cap I \Rightarrow \mathrm{udim}\, U = 1$

$\overset{n=1}{\Longrightarrow} \exists u \in U: \quad \mathrm{rann}(u) \cap U = 0$

Then $a := a' + u$ works:

$$\mathrm{rann}(a) \cap I \underset{\uparrow\, U \cap I' = 0}{=} \mathrm{rann}(a') \cap \mathrm{rann}(u) \cap I = \mathrm{rann}(u) \cap U = 0.$$

(3) „$\Rightarrow$" Let $I \leq_e R_R$, $a \in I$ as in (2) $\Rightarrow \mathrm{rann}(a) = 0$

$\overset{(1)}{\Longrightarrow} a \in R^\circ$

„$\Leftarrow$" Let $I \leq R_R$, $a \in I \cap R^\circ \overset{(1)}{\Longrightarrow} aR \leq_e R_R \Rightarrow I \leq_e R_R$ $\square$

Proof of Thm 5.5

(a) $\Rightarrow$ (b) (i) Q right noetherian, $Q = Q^r_{cl}(R)$

$\Rightarrow \mathrm{udim}\, R_R = \mathrm{udim}\, Q_Q < \infty$ (Prop. 3.12)

(ii) Q satisfies the ACC on right annihilators, and this descends to R:

Let $I_1 \subseteq I_2 \subseteq \ldots \subseteq R_R$ of right annihilators in R

$\Rightarrow \mathrm{lann}^R(I_1) \supseteq \mathrm{lann}^R(I_2) \supseteq \ldots$

$\Rightarrow \mathrm{rann}^Q(\mathrm{lann}^R(I_1)) \subseteq \mathrm{rann}^Q(\mathrm{lann}^R(I_2)) \subseteq \ldots$

so this chain stabilizes. But

$I_j = \mathrm{rann}^R(\mathrm{lann}^R(I_j)) = R \cap \mathrm{rann}^Q(\mathrm{lann}^R(I_j))$,

so the chain of I_j's stabilizes.

(iii) R semiprime: Let $I \leq_R R$ be a left ideal with $I^2 = 0$.

Claim: $I' = \mathrm{lann}(I) \leq_e R_R$

$[\forall r \in R \setminus I' \; \exists a \in I: \; ra \neq 0 \Rightarrow raI \subseteq I^2 = 0 \Rightarrow 0 \neq ra \in I' \cap rR]$

$\Rightarrow I'Q_Q \leq_e Q_Q$ (Prop 3.12)

$\overset{Q_Q \text{ semisimple}}{\Longrightarrow} I'Q_Q = Q_Q \Rightarrow \exists a' \in I', s \in R^\circ: \; 1 = a's^{-1}$

$\Rightarrow sI = a'I = 0 \overset{s \in Q^\times}{\Longrightarrow} I = 0$

(b) $\Rightarrow$ (c) By Corollary 4.9

(c) $\Rightarrow$ (d) Prop 5.6 (3)

(d) $\Rightarrow$ (a):

<u>R is right Ore:</u> Let $a \in R$, $s \in R^\circ \overset{(d)}{\Longrightarrow} sR \leq_e R_R$

$\Rightarrow$ „$a^{-1}(sR)$" $= \{r \in R : ar \in sR\} \leq_e R_R$ (Lemma 3.2(4))

$\overset{(d)}{\Longrightarrow} \exists s' \in R^\circ : as' \in sR \Rightarrow aR^\circ \cap sR \neq \emptyset$

So R is right Ore, we can form $Q = Q^r_{ce}(R)$.

<u>Q is semisimple:</u> Let $I \leq Q_Q$, $I^c := I \cap R \leq R_R$.

Let J be a complement to I^c in R_R.

$\Rightarrow I^c \oplus J \leq_e R_R \overset{(d)}{\Longrightarrow} \exists s \in R^\circ \cap (I^c \oplus J)$

Taking extensions, $(I^{ce} \oplus J^e) \cap Q^\times = (I \oplus J^e) \cap Q^\times \neq \emptyset$

$\Rightarrow Q = I \oplus J^e$

$\square$

<u>Remark 5.7</u> (1) (a) $\Rightarrow$ (b) shows that every right order in a right noetherian ring is right Goldie

(2) By Goldie's lemma, right regular elements in a semiprime right Goldie ring are regular. Left regular elements need not be regular!

<u>Cor 5.8</u> A ring R is a right order in a simple artinian ring
$\Leftrightarrow$ R is prime Goldie

<u>Proof.</u> Simple artinian rings are precisely the prime semisimple rings.

„$\Leftarrow$" $Q = Q^r_{ce}(R)$, Q is semisimple (Thm 5.5)

Let $I, J \leq Q_Q$ s.t. $IJ = 0$. Show: $I = 0$ or $J = 0$

$I^c J^c \subseteq IJ = 0 \Rightarrow I^c = 0$ or $J^c = 0$

$\rightarrow I = (I^c)^e = 0$ or $J = (J^c)^e = 0$

$\rightarrow Q$ prime $\Rightarrow Q$ simple artinian.

„$\Rightarrow$" Let R be a right order in the simple artinian ring Q

$\Rightarrow R$ is right Goldie (Thm 5.5)

Suppose $IJ=0$ $\quad (I, J \leq_R R)$ and $I \neq 0$.

$\Rightarrow QIQ = Q$

$\Rightarrow \exists x_i, y_i \in Q \ \exists a_i \in I$: $\quad 1 = \sum_{i=1}^{n} x_i a_i y_i \quad$, $y_i = r_i s^{-1}$, $r_i \in R$, $s \in S$

$\Rightarrow s = \sum_{i=1}^{n} \underbrace{(x_i a_i)}_{\in I} r_i \quad \Rightarrow \quad sJ \subseteq IJ = 0 \overset{s \in R^\circ}{\Rightarrow} J = 0$. $\square$

<u>Cor 5.9</u> (1) Every semiprime noetherian ring is an order in a semisimple ring

(2) Every noetherian prime ring is an order in a simple artinian ring.

(Goldie)
<u>Cor 5.10</u> (1) If R is semiprime [prime] right Goldie, then so is $M_n(R)$

(2) $R[x]$ is semiprime [prime] right Goldie $\Leftrightarrow$ R — " ————

<u>Proof</u>: We focus on the semiprime case, the prime case is similar

(1) R is a right order in $Q = Q^r_{cl}(R)$ with Q semisimple.

Then $M_n(R)$ is a right order in $M_n(Q)$, as

(i) If $(q_{ij}) \in M_n(Q)$, write $q_{ij} = r_{ij} s^{-1}$, $r_{ij} \in R$, $s \in R^\circ \cap Q^\times$ to get $(q_{ij}) = (r_{ij}) s^{-1}$

(ii) $R^\circ \subseteq Q^\times$: $\quad q \in R^\circ \Rightarrow \mathrm{rann}^Q(q) = 0 \quad [q \cdot a s^{-1} = 0 \overset{s \in Q^\times}{\Rightarrow} qa = 0 \Rightarrow a = 0]$

$\overset{\text{Prop 5.6(1)}}{\Longrightarrow} q \in Q^\circ \overset{Q \text{ quotient ring}}{\Longrightarrow} q \in Q^\times$.

Thm 5.5 implies the claim

(2) [Sketch] $\quad S := R[x]$,

Then ·) udim S_S = udim R_R $\quad$ ([LMR, 6.65])

·) R is right non-singular $\Leftrightarrow$ S right non singular [LMR, Ex. 7.35]

·) R semiprime $\Leftrightarrow$ S semiprime $\quad$ [FC, (10.18)]

Thm 5.5 implies the claim

Cor 5.11 A domain R is right Goldie iff it is right Ore.

Proof. „$\Rightarrow$" ✓ „$\Leftarrow$" Then $\operatorname{udim} R_R = 1 < \infty$ (Thm 2.8), & $0, R$ are the only annihilator ideals. $\square$

~~**Rem 5.12** •) If $R \subseteq Q$ is a right order, and Q is right Goldie, then so is R.~~ ~~[$\operatorname{udim} R_R = \operatorname{udim} Q_Q$, ACC on annihilators descends, see proof of Thm 5.5]~~

•) So, every right order in a right noetherian ring is right Goldie.

More Properties of (semiprime) right Goldie rings

Let R be a ring. For M_R, let

$$t(M) := \{ m \in M : \exists r \in R^\circ : mr = 0 \}$$

Prop 5.12 R is a semiprime right Goldie ring

$$\Longleftrightarrow t(M) = Z(M) \text{ for all (right) } R\text{-modules } M.$$

In this case,

$$t(M) = Z(M) = \ker(M_R \longrightarrow M \otimes_R Q)$$

where $Q = Q^r_{cl}(R)$

Proof: „$\Rightarrow$" $\operatorname{ann}(m) \leq R_R$ and by Thm 5.5(d),

$$m \in Z(M) \Leftrightarrow \operatorname{ann}(m) \leq_e R_R \Leftrightarrow \operatorname{ann}(m) \cap R^\circ \neq 0 \Leftrightarrow m \in t(M)$$

„$\Leftarrow$" Verify condition in T.5.5(d), namely that: $I \leq_e R_R \Leftrightarrow I \cap R^\circ \neq \emptyset$

•) Let $I \leq_e R_R \Rightarrow 1 + I \in Z(R/I) = t(R/I)$

$\Rightarrow \exists r \in R^\circ : (1+I) r \in I \Rightarrow r \in I$.

•) Let $I \leq R_R$ with $I \cap R^\circ \neq \emptyset$.

Let $r \in I \cap R^\circ \Rightarrow (1+I) r = 0 + I$ in R/I

$\Rightarrow 1 + I \in t(R/I) = Z(R/I) \Rightarrow I = \operatorname{ann}(1+I) \leq_e R_R$.

Finally, $M \otimes_R Q = MS^{-1}$ so the map has kernel $t(M)$. $\square$

Minimal Prime Ideals

Prop 5.13 Let R be a semiprime right Goldie ring.
Let $Q = Q^r_{cl}(R)$ with $Q = Q_1 \times \dots \times Q_t$ the Wedderburn decomposition (i.e., Q_i simple artinian).

$\forall i \in [1,t]$: $\quad M_i := \prod_{j \neq i} Q_j \trianglelefteq Q$ (if $t>1$), resp. $M_1 = 0$ if $t=1$.

Set $\quad P_i := M_i^c = M_i \cap R$.

Then $\{P_1, \dots, P_t\}$ are the t distinct minimal prime ideals of R.
Each R/P_i is right prime Goldie with $Q^r_{cl}(R/P_i) \cong Q_i$ and
$R \hookrightarrow R/P_1 \times \dots \times R/P_t$

Observations: (1) If $R_i \subseteq Q_i$ are rings $(i \in [1,t])$, then

$$R_1 \times \dots \times R_t \subseteq Q_1 \times \dots \times Q_t \text{ is a right order}$$
$$\Leftrightarrow \forall i \in [1,t]: \quad R_i \subseteq Q_i \text{ is a right order.}$$

(2) If $R \subseteq R' \subseteq Q$ are rings and $R \subseteq Q$ is a right order, then so is R':

[$(R')^\circ \subseteq Q^\times$: Let $r' = rs^{-1}$ with $r \in R$, $s \in R^\circ \subseteq Q^\times$
$\Rightarrow r = r's^{-1} \in (R')^\circ \cap R \subseteq R^\circ \Rightarrow r \in Q^\times \Rightarrow r' \in Q^\times$]

Now $Q = \{rs^{-1} : r \in R, s \in (R^\circ)\}$ $\quad s \in R^\circ \Rightarrow s \in Q^\times \cap R' \subseteq (R')^\circ$
$r \in R'$

Proof: P_i is prime: Since Q is right noetherian, for every $J \trianglelefteq R$, also $J^e \trianglelefteq Q$ (Prop 2.16(1))

Let $J, J' \trianglelefteq R$ s.t. $JJ' \subseteq P_i$

$\Rightarrow M_i \supseteq (JJ')^e = J^e (J')^e$

$\uparrow$ [L.M.R, (10.32)(2): „$\subseteq$" ✓ „$\supseteq$" $a_1 \underbrace{s_1^{-1} a_2 s_2^{-1}}_{\in (J')^e} = a_1 (\underset{\substack{\cap \\ J'}}{a'} \underset{\substack{\cap \\ R'}}{s^{-1}})$]

$\Rightarrow J^e \subseteq M_i$ or $(J')^e \subseteq M_i$

$\overset{\text{contract}}{\Longrightarrow} J \subseteq P_i$ or $J' \subseteq P_i$.

$\{P_1, \dots, P_t\}$ are all the minimal primes of R:

Since $P_i^e = M_i$, $P_i \nsubseteq P_j$ for $i \neq j$

Let $P \in \mathrm{Spec}(R)$

$\Rightarrow P_1 \cdots P_t \subseteq P_1 \cap \dots \cap P_t \subseteq M_1 \cap \dots \cap M_t = 0 \subseteq P$

$\overset{P \text{ prime}}{\Longrightarrow} \exists j: P_j \subseteq P$.

Quotient rings: Identify $Q_i = Q/M_i \Rightarrow R/P_i \subseteq Q/M_i = Q_i$

$P_1 \cap \dots \cap P_t = 0 \Rightarrow R \hookrightarrow R/P_1 \times \dots \times R/P_t \subseteq Q_1 \times \dots \times Q_t \subseteq Q$

$\overset{\text{Obs (4)}}{\Longrightarrow} R/P_1 \times \dots \times R/P_t$ is a right order in $Q_1 \times \dots \times Q_t$

$\overset{\text{Obs 1}}{\Longrightarrow} \forall i \in [1,t]$: R/P_i is a right order in Q_i

$\Rightarrow R/P_i$ prime right Goldie. $\square$

More facts about right ideals:

Let R be a semiprime right Goldie ring.

Lemma 5.14: Let $I \leq R_R$, $Q = Q^r_{cl}(R)$

(1) I is uniform $\Leftrightarrow$ IQ is a minimal (nonzero) right ideal

(2) If $I \leq R_R$ is uniform and $0 \neq x \in I$ then
 (i) $\mathrm{udim}(\mathrm{rann}\, x) = \mathrm{udim}\, R_R - 1$
 (ii) If $J \leq R_R$, $\mathrm{rann}_R(x) \subsetneq J$, then $J \leq_e R_R$

Proof. (1) $\mathrm{udim}_Q IQ = \mathrm{udim}_R I = 1$. Since Q is semisimple uniform right ideals are minimal.

(2) (i). $xQ = IQ$ is a minimal right ideal

$\overset{Q \text{ semisimple}}{\Longrightarrow} \mathrm{rann}_Q x$ is a maximal right ideal

$\Rightarrow \mathrm{udim}_Q(\mathrm{rann}_Q x) = \mathrm{udim}_Q Q_Q - 1 = \mathrm{udim}\, R_R - 1$

$\overset{\mathrm{rann}_R(x) = \mathrm{rann}_Q(x) \cap R}{\Longrightarrow} \mathrm{udim}_R(\mathrm{rann}_R(x)) = \mathrm{udim}\, R_R - 1$.

(ii) $\mathrm{rann}_R(x) = \mathrm{rann}_Q(x) \cap R$, $\mathrm{rann}_Q(x) = \mathrm{rann}_R(x)(R^\circ)^{-1}$

$\Rightarrow \mathrm{udim}_R(\mathrm{rann}_R(x)) = \mathrm{udim}_Q(\mathrm{rann}_Q(x)) = \mathrm{udim}\, Q_Q - 1$

$\overset{Q\text{ s.s.}}{\Longrightarrow}$ $\mathrm{rann}_Q(x)$ is a maximal ~~tot~~ right ideal of Q

$\Rightarrow JQ = Q$

$\Rightarrow JQ \leq_e Q \Rightarrow J \leq_e Q$ $\square$

Lemma 5.15 (i) $I \leq R_R$ essential $\Leftrightarrow IQ = Q$

(ii) A principal right ideal $aR \subseteq R$ is essential $\Leftrightarrow a \in R^\circ$

Proof: (i) $I \leq_e R \Leftrightarrow I \cap R^\circ \neq \emptyset \Leftrightarrow IQ \cap Q^\times \neq \emptyset$

(ii) „$\Rightarrow$" $aR \leq_e R_R \Rightarrow aQ = Q \Rightarrow a \in Q^\times \Longrightarrow a \in R^\circ$

„$\Leftarrow$": $aR_R \cong R_R$. $\square$

Prop 5.16 Let $I \leq_r R$ and $b \in R$. Then there exists $d \in I$

s.t. $\mathrm{udim}(b+d)R = \mathrm{udim}(bR + I)$

(Generalization of 5.6(3))

Proof: If $\mathrm{udim}(bR+I) = \mathrm{udim}(bR)$, take $d = 0$.

Otherwise $bR \subseteq bR + I$ is not essential, so there exists a uniform $U \subseteq I$ s.t. $bR \cap U = 0$.

Inductively, it suffices to find $u \in U$ s.t. $\mathrm{udim}(b+u)R = \mathrm{udim}\, bR \oplus U = \mathrm{udim}(bR) + 1$.

Let $A = \mathrm{rann}(U)$, $A' = \mathrm{lann}(A)$ $(A, A' \triangleleft R)$

Claim: (i) A' is the unique complement ideal of A in R

(ii) $\mathrm{rann}(b) \not\subseteq \mathrm{rann}(U)$

(i) $(A \cap \mathrm{lann}(A))^2 = 0$, $\overset{R\text{ semipri-}}{\Longrightarrow}$ $A \cap \mathrm{lann}(A) = 0$.

Suppose $A \cap C = 0$ for $C \triangleleft R \rightarrow CA \subseteq A \cap C = 0$

$\Rightarrow C \subseteq \mathrm{lann}(A) \subseteq A'$.

(ii) Suppose $\mathrm{rann}(b) \in \mathrm{rann}(U) = A$

$\Rightarrow A' \cong bA' \subseteq A'$ (bec. $A' \cap \mathrm{rann}(b) = 0$)

$\Rightarrow \mathrm{udim}(bA') = \mathrm{udim}(A') \Rightarrow bA' \leq_e A'_R$

$U \subseteq A' \Rightarrow 0 \neq bA' \cap U \subseteq bR \cap U$ $\lightning$. $\square$ (Claim)

Let $u \in U$ with $\mathrm{rann}(b) \subsetneq \mathrm{rann}(u)$. We show that $\mathrm{udim}(b+u)R = \mathrm{udim}(bR \oplus uR)$

$\mathrm{rann}(u) \subsetneq \mathrm{rann}(b) + \mathrm{rann}(u) \xRightarrow{5.14(2)} \mathrm{rann}(b) + \mathrm{rann}(u) \leq_e R_R$

$\xRightarrow{\text{sp. Goldie}} \exists c \in R^\circ : c \in \mathrm{rann}(b) + \mathrm{rann}(u)$

$\Rightarrow c = x + y$, $x \in \mathrm{rann}(b)$, $y \in \mathrm{rann}(u)$, so $bx = uy = 0$

$\Rightarrow bc = (b+u)y$, $uc = (b+u)x$

Let $br + us \in bR \oplus uR \xRightarrow{\text{right Ore}} \exists r', s' \in R, c' \in R^\circ$: $rc' = cr'$, $sc' = cs'$

Then $(br+us)c' = brc' + usc' = bcr' + ucs'$

$= (b+u)yr' + (b+u)xs' = (b+u)(yr' + xs') \in (b+u)R$

$\Rightarrow \mathrm{udim}\,(b+u)R = \mathrm{udim}\,(bR \oplus uR) = \mathrm{udim}\, bR + 1$

[Take $a_1 R \oplus \dots \oplus a_n R \subseteq bR \oplus uR$

$\Rightarrow \exists c' \in R^\circ : a_i c' \in (b+u)R$

$a_1 c' R \oplus \dots \oplus a_n c' R \subseteq (b+u)R$]

Cor. 5.17 Every essential right ideal I of R is gen. by regular elements

Proof. $\exists c \in R^\circ \cap I$ since R is semiprime right Goldie

$\Rightarrow \mathrm{udim}(cR_R) = \mathrm{udim}(R_R)$

For every $b \in I$, $\exists d \in cR$: $\mathrm{udim}\,(b+d)R = \mathrm{udim}\,(bR + cR)$

$= \mathrm{udim}(R) = \mathrm{udim}(I)$.

$\Rightarrow (b+d)R \leq_e R \Longrightarrow \underbrace{b+d}_{} \in R^\circ$ (Prop 5.17)

But $b \in (b+d)R + bR$ $\square$

6. Reduced Rank

6.1 (Reminder on prime radical) 6.1 Definition

In a ring R the prime radical (lower nilradical) N is the intersection of all prime ideals. As intersection of prime ideals it is semiprime, and it is the smallest semiprime ideal.

We need to facts for noetherian rings

Prop 6.1 Let R be a right noetherian ring

(1) There are finitely many minimal prime ideals

(2) The prime radical is nilpotent

Def 6.2 Let R be a semiprime right Goldie ring, M_R an R-module. The reduced rank of M is

$$\rho_R(M) = \operatorname{udim}\left(M/t(M)\right) \in \mathbb{N}_0 \cup \{\infty\}$$

Remark: 1) If $Q = Q^r_{cl}(R)$, then

$$\rho_R(M) = \operatorname{length}\left(\underbrace{A \otimes_R Q}_{\text{semisimple } Q\text{-module}}\right)$$

2) If M is f.g., then so is $M \otimes_R Q$, and thus $\rho_R(M) < \infty$

To extend the definition to noetherian rings, we need the Schreier refinement theorem.

Def 6.3 Let R be a ring, M_R R-module. •) A submodule series is a finite chain of submodules

$$A_0 = 0 \leq A_1 \leq \dots \leq A_k = M$$

•) A refinement is a submodule series that includes all M_i

•) Two submodule series $A_0 = 0 \leq A_1 \leq \dots \leq A_k' = M$,

$B_0 = 0 \leq B_1 \leq \dots \leq B_e \leq M$

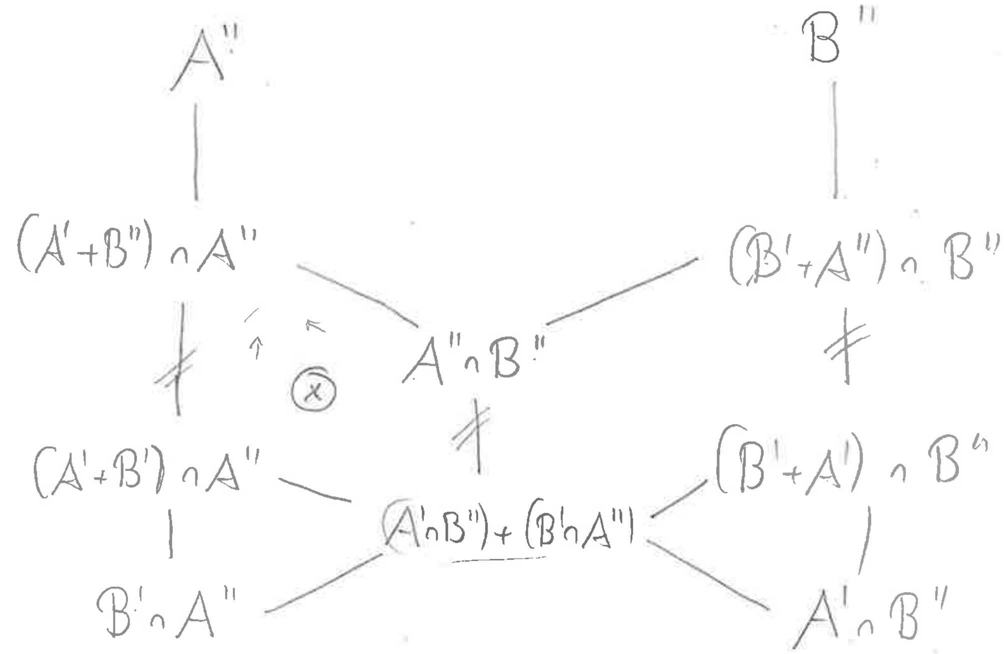
are equivalent (isomorphic) if $k=l$, and there exists a permutation $\sigma \in S_k$ s.t.

$$A_i / A_{i-1} \cong B_{\sigma(i)} / B_{\sigma(i)-1} \quad \forall i.$$

Lemma 6.4 (Zassenhaus/Butterfly lemma). R ring, M_R

$$A', A'', B', B'' \leq M \quad \text{s.t.} \quad A' \leq A'', \quad B' \leq B''$$

Then
$$\frac{(A'+B'')\cap A''}{(A'+B')\cap A''} \cong \frac{(B'+A'')\cap B''}{(B'+A')\cap B''}$$



Proof:
$$(A'+B'')\cap A'' = A' + (B''\cap A'') \quad (\text{modularity law, } A' \subseteq A'')$$
$$(A'+B')\cap A'' = A' + (B'\cap A'')$$

$\overset{\text{in }(*)}{\Longrightarrow}$
$$(A''\cap B'') + ((A'+B')\cap A'') = (A''\cap B'') + (A' + \cancel{(B'\cap A'')})$$
$$= (A''\cap B'') + A' = (A'+B'')\cap A''$$

and
$$(A''\cap B'')\cap((A'+B')\cap A'') = (A''\cap B'')\cap(A' + (B'\cap A''))$$
$$= (A'\cap \cancel{A''}\cap B'' + B'\cap A'')$$

$$\Longrightarrow \frac{(A'+B'')\cap A''}{(A'+B')\cap A''} \cong \frac{A''\cap B''}{(A'\cap B'')+(B'\cap A'')}$$

+ symmetry. □

Theorem 6.5 (Schreier) R ring, M_R module

Any two submodule series for M have isomorphic refinements.

Proof. Let $0 = A_0 \leq A_1 \leq \ldots \leq A_k = M$,

$0 = B_0 \leq B_1 \leq \ldots \leq B_e = M$

be submodule series.

Set $A_{ij} := (A_i + B_j) \cap A_{i+1}$ $(0 \leq i \leq k-1,\ 0 \leq j \leq l)$

$B_{ji} = (B_j + A_i) \cap B_{j+1}$ $(0 \leq j \leq l-1,\ 0 \leq i \leq k)$

These give refinements of the initial series $(A_i \leq A_{ij} \leq A_{i+1}$, $B_j \leq B_{ji} \leq B_{j+1})$

Zassenhaus lemma $\Rightarrow$ $A_{i,j+1}/A_{ij} \cong B_{j,i+1}/B_{ji}$

$\square$

Prop 6.6: Let R be a right noetherian ring, N the prime radical, M_R an R-module. Let

$M = A_0 \geq A_1 \geq \ldots \geq A_m = 0$ and $M = B_0 \geq B_1 \geq \ldots \geq B_n = 0$

be submodule series s.t. $(A_i/A_{i+1})N = 0$ and $(B_i/B_{i+1}) = 0$

for all $i \in [0, m-1]$, $j \in [0, n-1]$

Then $$\sum_{i=0}^{m-1} S_{R/N}(A_i/A_{i+1}) = \sum_{j=0}^{n-1} S_R(B_i/B_{i+1})$$

Proof: Case 1 $MN = 0$,

Then M is a R/N-module with R/N semiprime right Goldie.

Let $S := (R/N)^\circ$. For every R/N-module K, $K \otimes_{R/N} (R/N)S^{-1} \cong KS^{-1}$

and hence $S_{R/N}(K) = \text{length}(KS^{-1})$.

Now

$$MS^{-1} = A_0 S^{-1} \geq A_1 S^{-1} \geq \dots \geq A_m S^{-1} = 0$$

and $A_i S^{-1} / A_{i+1} S^{-1} \cong \left(A_i / A_{i+1} \right) S^{-1}$. (as $(R/N) S^{-1}$-modules)

Thus

$$\sum_{i=0}^{m-1} \rho_{R/N}(A_i/A_{i+1}) = \sum_{i=0}^{m-1} \operatorname{length}\left((A_i/A_{i+1}) S^{-1} \right) = \sum_{i=0}^{m-1} \operatorname{length}\left(A_i S^{-1} / A_{i+1} S^{-1} \right)$$

$$= \operatorname{length}(MS^{-1}).$$

Similarly $\sum_{i=0}^{n-1} \rho_{R/N}(B_i/B_{i+1}) = \operatorname{length}(MS^{-1})$.

Case 2: General Case. By Case 1, in particular, for R/N-modules, the sums remain invariant under refinement of the chains. By the Schreier refinement theorem, the chains have equivalent refinements. So wlog $\exists$ permutation σ s.t. $n = m$ an $B_i/B_{i+1} \cong A_{\sigma(i)}/A_{\sigma(i)+1}$ for $i \in [0, n-1]$. Then clearly the sums are the same. $\square$

Def 6.7 Let R be a right noetherian ring w. prime radical N and M a (right) R-module. Let

$$M = M_0 \geq M_1 \geq \dots \geq M_n = 0$$

be a submodule series s.t. $(M_i/M_{i+1}) N = 0$ for all $i \in [0, n-1]$ ($M_i = MN^i$, $N^n = 0$ works, because N is nilpotent)

The reduced rank of M is

$$\rho_R(M) := \rho(M) := \sum_{i=0}^{n-1} \rho_{R/N}(M_i/M_{i+1}).$$

<u>Remarks</u>: If R is semiprime, this Def. agrees with Def. 6.2 by Prop. 6.6

<u>Example</u>: ·) $\rho_{\mathbb{Z}/p^n\mathbb{Z}}(\mathbb{Z}/p^e\mathbb{Z}) = e$ $(p \in \mathbb{P}, e \leq n)$, $N_{\mathbb{Z}/p^n\mathbb{Z}} = p\mathbb{Z}/p^n\mathbb{Z}$

·) $\rho_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}) = 0$ $(n \geq 1)$, bec. $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$

<u>Lemma 6.8</u> Let R be a right noetherian ring, M_R a right module

(1) If M_R is f.g., then $\rho_R(M) < \infty$

(2) If $K \leq M$, then $\rho(M) = \rho(K) + \rho(M/K)$, i.e., $\rho_R(-)$ is additive on short exact sequences.

(3) If $\rho(M) < \infty$ and $K \leq M$, then $\rho(K) = \rho(M) \Leftrightarrow \rho(M/K) = 0$

<u>Proof</u>: (1) ✓ (2) Refine $0 \leq K \leq M$

(3) by (2) □

For $I \triangleleft R$, let $\mathcal{C}(I) = \{r \in R : r + I \in (R/I)^{\circ}\}$, $\mathcal{C}(0) = R^{\circ}$

<u>Lemma 6.9</u> Let R be a right noetherian ring with prime radical N, M_R a right module.

Then M_R is $\mathcal{C}(N)$-torsion (i.e. $\forall m \in M \, \exists r \in \mathcal{C}(N): mr = 0$) $\Leftrightarrow$ $\rho_R(M_R) = 0$

<u>Proof</u>: Let $M = M_0 \geq M_1 \geq \ldots \geq M_n = 0$ be a submodule series with $(M_i/M_{i+1})N = 0$ for all $i \in [0, n-1]$

$\rho_R(M) = 0 \Leftrightarrow \forall i \in [0, n-1]: \rho_{R/N}(M_i/M_{i+1}) = 0$

$\Leftrightarrow \forall i \in [0, n-1]: M_i/M_{i+1}$ torsion (R/N)-module

$\Leftrightarrow$ — " — M_i/M_{i+1} $\mathcal{C}(N)$-torsion

$\Leftrightarrow M$ $\mathcal{C}(N)$-torsion □

7. The Principal Ideal Theorem

Let R be a ring. The <u>height</u> of a prime ideal P is

$$ht(P) = \sup\{n \geq 0 : \exists \text{ prime ideals } P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n = P\}$$

~~7.1 Krulls P.I.T.~~

We first recall Krull's Principal Ideal Theorem, following a proof of Rees [Kaplansky, „Commutative Rings", '74 – Thm. 142]

<u>Recall:</u> For a commutative ring R TFAE:

.) R is artinian

.) R has Krull dimension 0 (noetherian)

.) Every f.g. module over R has finite length

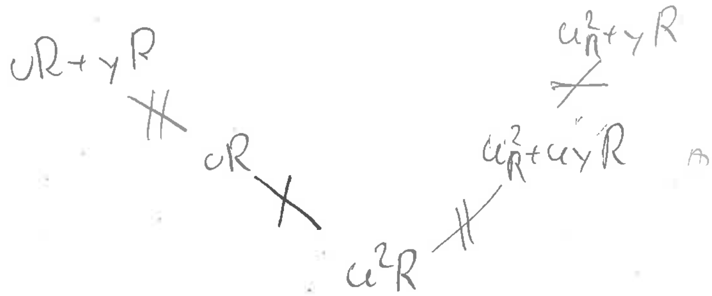
<u>Thm 7.1</u> Let R be a commutative noetherian ring, $x \in R \setminus R^\times$, and P a prime ideal minimal over x. Then $ht(P) \leq 1$.

<u>Prop 7.2</u> Let R be a commutative domain, $0 \neq u, y \in R$.

Then (1) $uR+yR/uR \cong u^2R+uyR/u^2R$

(2) Assume $tu^2 \in yR \Rightarrow tu \in yR$ $(\forall t)$

Then $uR/u^2R \cong u^2R+yR / u^2R+uyR$.



Proof: (1) $uR+yR \xrightarrow{\sim} u^2R+uyR$ induces the claimed iso

(2) uR/u^2R is cyclic, $\operatorname{ann}\left(uR/u^2R\right) = uR$, i.e. $uR/u^2R \cong R/uR$

$M = u^2R+yR/u^2R+uyR$ is cyclic (generated by $y+u^2+uyR$).

Suffices to show: $\operatorname{ann}\left(u^2R+yR/u^2R+uyR\right) = uR$, then $M \cong R/uR$

„$\supseteq$" ✓ „$\subseteq$" Let $r \in R$ s.t. $yr = u^2s+uyt$ (*) $(s,t\in R)$.

$(*) \Rightarrow u^2s \in yR \xRightarrow{\text{Assumption}} us \in yR \Rightarrow us = yw$, $w \in R$

$\xRightarrow{(*)} \cancel{y}r = u\cancel{y}w + u\cancel{y}t \Rightarrow r \in uR$ □

Proof of Thm 7.1 By contradiction.

Assume there are prime ideals $P_1 \subsetneq P_2 \subsetneq P$.

$$\begin{tikzcd}[row sep=small] & R \arrow[d, dash] \\ & P \arrow[d, dash] \arrow[dl, dash] \\ xR & P_2 \arrow[d, dash] \\ & P_1 \end{tikzcd}$$

Two reductions: (1) Pass to R/P_1 ($\Rightarrow P_1 = 0$)

(2) Localize R/P_1 at P/P_1 ($\rightarrow P/P_1$ maximal)

After this reset notation: R is a local domain with maximal ideal M, $x \in R\setminus\{0\}$, M minimal prime over xR, $0 \subsetneq Q \subsetneq M$ prime ideal

$$\begin{tikzcd}[row sep=small] & R \text{ (local)} \arrow[d, dash] \\ & M \text{ max} \arrow[d, dash] \arrow[dl, dash] \\ xR & Q \arrow[d, dash] \\ & 0 \end{tikzcd}$$

Let $y \in Q\setminus\{0\}$,

$I_k := \{t \in R : tx^k \in yR\} = (yR : x^kR) \cap R$

$\rightarrow I_1 \subseteq I_2 \subseteq \dots$ becomes stable, say at I_n

$\rightarrow [tx^{2n} \in yR \Rightarrow tx^n \in yR]$

$u := x^n$, so: $tu^2 \in yR \Rightarrow tu \in yR$

$T := R/u^2R$ has as only prime ideal $M/u^2R \Rightarrow$ Every f.g. T-module has finite length.

$\xRightarrow{\text{P. 7.2}} \operatorname{length}\left(uR+yR/u^2R\right) = \operatorname{length}\left(u^2R+yR/u^2R\right)$

But $u^2R+yR \subseteq uR+yR$, so $u^2R+yR = uR+yR$.

$\Rightarrow$ $u = u^2s+yt$ $(s,t \in R)$

$\Rightarrow$ $\underbrace{u(1-su)}_{\in R^\times\ (u \notin R^\times,\ R \text{ local})} = yt$ $\Rightarrow$ $u \in yR \subseteq Q$ $\lightning$ M minimal over xR (here one $uR = x^nR$)

$\square$

7.2 Jategaonkar's P.I.T

Example: $R = M_2(\mathbb{Z})$

prime ideals: 0, $M_2(p\mathbb{Z})$, p prime

So $\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}$, is not contained in any (prime) ideal.

•) An element $x \in R$ is normal if $xR = Rx$

Theorem 7.3 (Jategaonkar '74, Chatters-Goldie-Hajarnavis-Lenagan '79)
Let R be a right noetherian ring, $x \in R \setminus R^\times$ normal and P a prime ideal minimal over $xR = Rx$. Then $ht(P) \leq 1$.

We prove more generally,

Thm 7.4 (CGHL '79) Let R be a right noetherian ring and $X \trianglelefteq R$, $X \neq R$ an invertible ideal. If $P \triangleleft R$ is a prime ideal minimal over X, then $ht(P) \leq 1$

Here X is invertible if there is an overring $T \supseteq R$ and a R-R-bimodule $X^{-1} \subseteq T$ s.t. $XX^{-1} = X^{-1}X = R$.

Fix a noetherian ring R, and $T \supseteq R$ an overring.
We can take $T := \bigcup_{n \geq 0} X^{-n}$

Lemma 7.5 Let $A, B \leq T_R$ be right R-submodules, $X \triangleleft R$ invertible. Then

(1) $(A \cap B)X = AX \cap BX$

(2) $(A \cap B)X^{-1} = AX^{-1} \cap BX^{-1}$

(3) If $P \triangleleft R$ is a prime ideal s.t. $X \nsubseteq P$, then

(i) $XP = P \cap X = PX$ and $X^{-1}P = PX^{-1}$

(ii) $X+P/P$ is an invertible ideal of R/P

Proof: (1) „$\subseteq$": $(A \cap B)X \subseteq AX \cap BX$

„$\supseteq$": $(AX \cap BX)X^{-1} \subseteq AXX^{-1} \cap BXX^{-1} \subseteq A \cap B$

$\Rightarrow AX \cap BX = (AX \cap BX)X^{-1}X \subseteq (A \cap B)X$

(2) Analogous

(3) (i) We show $PX = P \cap X$; $XP = P \cap X$ follows similarly.

„$\subseteq$": $PX \subseteq P \cap X$ ✓

„$\supseteq$": $(P \cap X)X^{-1} \subseteq XX^{-1} \subseteq R \Rightarrow (P \cap X)X^{-1} \triangleleft R$

$(P \cap X)X^{-1} \cdot X = P \cap X \subseteq P \xRightarrow{P \text{ prime}} {}^{X \nsubseteq P} (P \cap X)X^{-1} \subseteq P$

$\Rightarrow (P \cap X) \subseteq P$

Now $PX = XP \Rightarrow P\underbrace{XX^{-1}}_{R} \subseteq XPX^{-1} \Rightarrow P \subseteq XPX^{-1}$

$\Rightarrow X^{-1}P \subseteq X^{-1}XPX^{-1} = PX^{-1}$ and similarly $PX^{-1} \subseteq X^{-1}P$

(ii) Take $T = \bigcup_{n \geq 0} X^{-n}$. X^n, $n \geq 1$ is invertible and $X^n \nsubseteq P$.

$\xRightarrow{(i)} PX^n = P \cap X^n$ and $PX^{-n} = X^{-n}P$

$\xRightarrow{(2)} \underline{P} = PX^nX^{-n} = (P \cap X^n)X^{-n} \overset{!}{=} \underline{PX^{-n} \cap R}$

$\Rightarrow \underline{PT \cap R} = \left(\bigcup_{n \geq 0} PX^{-n}\right) \cap R = \bigcup_{n \geq 0} (PX^{-n} \cap R) = \underline{P.}$

$\underline{PT} = \bigcup_{n \geq 0} PX^{-n} = \bigcup_{n \geq 0} X^{-n}P = \underline{TP} \Rightarrow PT \triangleleft T$

$$\begin{cases} R \longrightarrow T/PT \\ r \longmapsto r+PT \end{cases}$$ has kernel $PT \cap R = R$,

so $R/P \hookrightarrow T/PT$. $X+P/P$ has inverse $X^{-1}+PT/PT$. $\square$

<u>Lemma 7.6</u> If $I \leq R_R$, then $\exists n \geq 1: \; I \cap X^{2n} \subseteq IX^n$.

<u>Proof:</u> ACC of right ideals:

$$\underset{IX^{-1}\cap R}{\underset{\parallel}{(I\cap X)X^{-1}}} \subseteq \underset{IX^{-2}\cap R}{\underset{\parallel}{(I\cap X^2)X^{-2}}} \subseteq \ldots \subseteq R \quad \text{(Lemma 7.5(2))}$$

becomes stationary at some $n \geq 1$, so $(I\cap X^n)X^{-n} = (I \cap X^{n+1})X^{-n-1} = \ldots$

$$\Longrightarrow (I\cap X^n)X^{-n} = (I\cap X^{2n})X^{-2n}$$

$$\Longrightarrow (I \cap X^{2n}) = (I\cap X^n)X^n \subseteq IX^n.$$ $\square$

<u>Lemma 7.7</u> Let R be a semiprime right Goldie ring (not necessarily noetherian). Let $P_1, \ldots, P_n$ be the minimal prime ideals of R. Then $R^\circ = \mathcal{C}(0) = \mathcal{C}(P_1) \cap \ldots \cap \mathcal{C}(P_n)$

<u>In particular:</u> If R is a right noetherian ring, $N \underset{\neq}{\trianglelefteq} R$ semiprime and $P_1, \ldots, P_n$ are the prime ideals minimal over N, then $\mathcal{C}(N) = \mathcal{C}(P_1) \cap \ldots \cap \mathcal{C}(P_n)$

<u>Proof</u> (Using Prop 5.13) $Q = Q^r_{cl}(R)$, $Q_i := Q^r_{cl}(R/P_i)$

$$\begin{tikzcd} Q \arrow[r, phantom, "\cong"] & Q_1 \times \ldots \times Q_n \\ R \arrow[u, hook] \arrow[r, hook] & R/P_1 \times \ldots \times R/P_n \arrow[u, hook] \end{tikzcd}$$

$$\begin{aligned} x \in R^\circ &\Leftrightarrow \overset{=(x_1,\ldots,x_n)}{x} \in Q^\times \\ &\Leftrightarrow \forall i: \; x_i \in Q_i^\times \\ &\Leftrightarrow \forall i: \; x_i \in (R/P_i)^\circ \\ &\Leftrightarrow \forall i: \; x_i \in \mathcal{C}(P_i). \end{aligned}$$

$\square$

Prop 7.8 Let $X \triangleleft R$ invertible, $T := \bigcup_{n \geq 0} X^{-n}$.
If $B \subseteq A$ are f.g. R-submodules of T_R, then
$$\rho_{R/X}(A/B) = \rho_{R/X}(AX/BX)$$

Proof: Let $N \triangleleft R$ s.t. $X \subseteq N$ and N/X is the prime radical of R/X.
$R \supseteq X^{-1}\underbrace{NX}_{\subseteq X}$ is nilpotent modulo X, so $X^{-1}NX \subseteq N$
$\Rightarrow NX \subseteq XN$ and symmetrically $XN \subseteq NX$
$\Rightarrow NX = XN$.
It suffices to show $\rho_{R/X}(A/B) = \rho_{R/X}(AX/BX)$ when $AN \subseteq B$ ($\Rightarrow AXN \subseteq BXY$)
The general claim then follows by taking a suitable submodule series in $B \subseteq A$.

There is a natural lattice iso
$$\{\text{submodules of } A/B\} \leftrightarrow \{\text{submodules of } AX/BX\}$$
$$C/B \leftrightarrow CX/BX$$
Show that $\mathcal{C}(N)$-torsion submodules correspond to each other under this iso, then $\rho_{R/X}(A/B) = \rho_{R/X}(AX/BX)$

Show: If A/B is a torsion R/N-module, then so is AY/BY
where $Y = X$ or $Y = X^{-1}$.
$YN = NY \overset{(*)}{\Rightarrow} \forall t \in T: tY \subseteq N \Leftrightarrow Yt \subseteq N$

Let $a \in AY$, $K := \{r \in R : ar \in BY\} \supseteq N$.
Show $K \cap \mathcal{C}(N) \neq \emptyset$, by showing that $K/N \leq_e R/N$.
(since R/N is semiprime right Goldie this suffices).
Let $I \leq R_R$ s.t. $N \subseteq I$ and $I \cap K = N$
Let $s \in IY^{-1} \subseteq Y^{-1} \Rightarrow as \in AYY^{-1} = A \overset{A/B\ \mathcal{C}(N)\text{-torsion}}{\Longrightarrow} \exists c \in \mathcal{C}(N)$:
$asc \in B \Rightarrow ascY \subseteq BY$

and $sc\,Y \subseteq \underbrace{IY^{-1}}_{s}\underbrace{R}_{c}Y \subseteq I \subseteq R$

$\Longrightarrow sc\,Y \subseteq I \cap K = N$ (Def. of K)

$\overset{(*)}{\Longrightarrow} Ysc \subseteq N \Longrightarrow Ys \subseteq YIY^{-1} \subseteq R \overset{c \in \mathcal{C}(N)}{\Longrightarrow} Ys \subseteq N$

$\overset{(*)}{\Longrightarrow} sY \subseteq N$

s arbitrary in $IY^{-1} \Longrightarrow I = (IY^{-1})Y \subseteq N \Longrightarrow K/N \leq_e R/N$.

$\square$

$$\begin{tikzcd}[row sep=small] & P \arrow[dl, dash, "\text{minimal}"'] \arrow[d, dash] \\ X & P_2 \arrow[d, dash] \\ & P_1 \end{tikzcd}$$

Proof of Thm 7.4 Suppose $ht(P) \geq 2$.

Let $P_1 \subsetneq P_2 \subsetneq P$ be a chain of prime ideals

Pass to R/P_1: $X + P_1/P_1$ is invertible (lemma 7.5(3)(ii)) in R/P_1, P/P_1 minimal over X/P_1, $P_1/P_1 = 0$

Reset notation: $0 \subsetneq Q \subsetneq P$ prime ideals of R, P min. $/X$

$$\begin{tikzcd}[row sep=small] & P \arrow[dl, dash] \arrow[d, dash] \\ X & Q \arrow[d, dash] \\ & 0 \end{tikzcd}$$

Let $y \in Q \cap R^\circ$.

$\overset{\text{lemma}}{\Longrightarrow} \exists n \geq 1: \; yR \cap X^{2n} \subseteq yX^n$

P is minimal also over X^n, so, replace X by X^n wlog $n=1$, so $yR \cap X^2 \subseteq yX$

Modularity $\Longrightarrow yR \cap (X^2 + yX) = (yR \cap X^2) + yX = yX$

Let $N \supseteq X$ be the ideal of R s.t. N/X is the prime radical of R/X, $\mathcal{S} = \mathcal{S}_{R/X}$

$$\begin{tikzcd}[column sep=small] X + yR \arrow[d, dash] \arrow[drr, dash] & & \\ X \arrow[dr, dash] & & X^2 + yR \arrow[dl, dash] \\ & X^2 + yX & \end{tikzcd}$$

$$\delta\left(\frac{X^2+yR}{X^2+yX}\right) = \delta\left(\frac{X^2+yR+yX}{X^2+yX}\right)$$

$$= \delta\left(\frac{yR}{yR\cap(X^2+yX)}\right) = \delta\left(\frac{yR}{yX}\right) \overset{y\in R^\circ}{=} \delta\left(R/X\right)$$

$$\begin{tikzcd}[column sep=small] & X^2+yR+yR \arrow[dl, dash] \arrow[dr, dash] & \\ yR \arrow[dr, dash] & & X^2+yX \arrow[dl, dash] \\ & yR\cap(X^2+yX) & \end{tikzcd}$$

$$\delta\left(\frac{X+yR}{X^2+yX}\right) = \delta\left(\frac{X+yR}{X}\right) + \delta\left(\frac{X}{X^2+yX}\right) =$$

$$\overset{P78}{=} \delta\left(\frac{X+yR}{X}\right) + \delta\left(\frac{R}{X+yR}\right) = \delta(R/X)$$

__So__ $\delta_{R/X}\left(\frac{X+yR}{X^2+yR}\right) = 0$ (additivity of δ on SES)

$\Rightarrow$ $\frac{X+yR}{X^2+yR}$ is $\mathcal{C}(N/X)$-torsion as R/X-module; (Lemma 6.9)

hence it is $\mathcal{C}(N)$-torsion as R-module

Let $x\in X \Rightarrow \exists\, c\in\mathcal{C}(N)$: $xc\in X^2+yR \subseteq X^2+Q$

$\Rightarrow xc \in X^2+(X\cap Q) = X^2+XQ$ (Lemma 7.5(3)(i))

$\Rightarrow X^{-1}xc \subseteq X+Q \subseteq P$

However, $X^{-1}x\subseteq R$, $c\in\mathcal{C}(N)\subseteq\mathcal{C}(P)$ (Lemma 7.7), as P/N is a minimal prime of the semiprime right Goldie ring R/N.

$\Rightarrow X^{-1}x\subseteq P \Rightarrow R=X^{-1}X\subseteq P$ ↯. $\square$

__Def. 7.9__ Let R be a ring.

.) R is __right bounded__ if every essential right ideal of R contains a ~~nonzero~~ two-sided ideal that is essential as right ideal.

.) R is __right fully bounded__ if every prime factor ring is right bounded.

.) A __right FBN__ ring is a right fully bounded ~~noeth~~ right noetherian ring.

Exm.) If R is a module-finite algebra / a comm. ring S, then R is FBN.
If R is prime & S is noetherian, then every essential left or right ideal of R contains a nonzero central element (Formanek condition)

.) The same holds for prime PI rings.

Proof of Thm 7.3: Wrl R is prime (as in prev proof)
$\Rightarrow$ c regular, $Q^r_{cl}(R)$ exists and contains c^{-1}
$\Rightarrow$ cR is invertible $\xrightarrow{\text{Thm } 7.4}$ Thm 7.3

Theorem 7.10 Let R be a prime right FBN ring s.t. every essential right ideal contains a $\neq 0$ central element.
Let $x \in R^{\circ} \setminus R^{\times}$ and $B = \operatorname{ann}(R/xR) \triangleleft R$ (the largest ideal contained in xR). Let $P \triangleleft R$ be a prime minimal over B and assume $x \notin \mathcal{C}(P)$. Then $h(P) \leq 1$

Proof: (Modification of Thm 7.4) Instead of X consider xR,
let $y \in Q$ be nonzero central.
R/yR right noeth $\Rightarrow \exists n \geq 1: \forall r \in R: \ x^{n+1} r \in yR \rightarrow x^n r \in yR$
(cf. Proof of Thm 7.1).
Replace x by x^n: $\quad yR \cap (x^2R + yxR) = yxR$

$\rho = \rho_{R/B}$

$$\rightarrow \rho\left(\frac{x^2R + yR}{x^2R + yxR}\right) = \rho\left(\frac{x^2R + yR + yxR}{x^2 + yxR}\right)$$

$$= \rho\left(\frac{yR}{yR \cap (x^2R + yxR)}\right) = \rho\left(\frac{yR}{yxR}\right) = \rho(R/xR)$$

$$\begin{array}{c} R \\ | \\ xR+yR \\ | \\ xR \\ | \\ x^2R+yxR \end{array}$$

R/xR

$$S\left(\frac{xR+yR}{x^2R+yxR}\right) = S\left(R/xR\right):$$

$$S\left(\frac{xR}{x^2R+yxR}\right) = S\left(\frac{xR}{x^2R+\underline{\underline{xy}}R}\right) = S\left(\frac{R}{xR+yR}\right)$$

$\Longrightarrow \exists c \in \mathcal{C}(N)$: ($N/B \trianglelefteq R/B$ prime radical)

s.t. $xc \in x^2R+Q$, $xR \supseteq B$, $B \not\subseteq Q$

$\Longrightarrow x \in \mathcal{C}(Q) \Longrightarrow c \in xR+Q \subseteq xR+P$

$c \in \mathcal{C}(N) \subseteq \mathcal{C}(P) \overset{}{\underset{(*)}{\Longrightarrow}} x \in \mathcal{C}(P)$ ↯. □

$x \in \mathcal{C}(P)$ can occur:

$R = \begin{pmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$, $x = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{pmatrix}$,

$P = \begin{pmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$, $x \in \mathcal{C}(P)$ (but note $ht(P) \leq 1$)

Question Does 7.10 hold w/o the Formanek condition?

(∗) x is a left divisor of c, $c = xx'$

$\Longrightarrow 1 = x(x'c')$ for some $c' \in Q^\times$

$\overset{\text{noeth} \Rightarrow \text{Dedekind-fin}}{\Longrightarrow} (x'c')x = 1 \Longrightarrow x \in Q^\times$

8. Bounded Factorizations

8.1 Basic facts

Let H be a monoid. $u \in H \setminus H^{\times}$ is an atom (irreducible) if $u = ab$ $(a, b \in H) \Rightarrow a \in H^{\times}$ or $b \in H^{\times}$.

For $a \in H \setminus H^{\times}$, let

$$L(a) := \{ k \in \mathbb{N}_0 : \exists \text{ atoms } u_1, \dots, u_k : \ a = u_1 \cdots u_k \}$$

and set $L(\varepsilon) = \{0\}$ for $\varepsilon \in H^{\times}$.

<u>Def 8.1</u> Let H be a monoid.

(1) H is <u>atomic</u> if $L(a) \neq \emptyset$ for all $a \in H$

(2) H has <u>BF (bounded factorizations)</u> if

$$1 < |L(a)| < \infty \quad \text{for all } a \in H$$

(3) H is <u>half-factorial</u> if $|L(a)| = 1$ for all $a \in H$.

A monoid H is <u>unit-cancellative</u> if $a = au$ or $a = ua$ with $u \in H$ implies $u \in H^{\times}$. In such a monoid every left [right] invertible element is invertible: $1 = uv \Rightarrow u = u(vu)$

$[vu = \varepsilon \Rightarrow uvu = u\varepsilon = u] \xRightarrow{\text{unit}} vu = 1$

In this case $\sup L(ab) \geq \sup L(a) + \sup L(b)$ $\forall a, b \in H$

<u>Def. 8.2</u> Let H be a monoid, $\lambda : H \to \mathbb{Z}_{\geq 0}$

(1) λ is a <u>right length function</u> if $\lambda(a) > \lambda(b)$ whenever $a = bc$, $a, b, c \in H$, $c \notin H^{\times}$

(2) λ is a <u>length function</u> if $\lambda(a) > \lambda(b)$ whenever $a = dbc$ with $c, d \in H$ not both units

(3) λ is a <u>superadditive length function</u> if

(i) $\lambda(ab) \geq \lambda(a) + \lambda(b)$ for $a, b \in H$

(ii) $\lambda(a) = 0 \Rightarrow a \in H^{\times}$

Lemma 8.3: Let H be a monoid with right length function λ

(1) If $a_1, \dots, a_k \in H \setminus H^\times$, then $\lambda(a_1 \cdots a_k) \geq k$ $(k \geq 0)$.

In particular: $a \in H \setminus H^\times \Longrightarrow \lambda(a) > 0$

(2) H is unit-cancellative

Proof: (1) Induction on k.

$k=0$: $1 = a_1 \cdots a_k$ (empty product) ✓

$k \geq 1$: $\lambda(a_1 \cdots a_k) > \lambda(a_1 \cdots a_{k-1}) \geq k-1$.

(2) (i) Let $a = ab$ with $b \in H \setminus H^\times \Longrightarrow \lambda(a) > \lambda(a)$ ↯

(ii) Let $a = ba$, $b \in H \setminus H^\times$.

Then $a \in H \setminus H^\times$ [otherwise: $b = aa^{-1} = 1$], $a = b^k a$ for all $k \geq 0$.

$\Longrightarrow \forall k \geq 0$: $\lambda(a) \geq \lambda(b^k) \geq k$ ↯. □

Theorem 8.4 Let H be a monoid. TFAE

(a) H has a superadditive length function

(b) H has a length function

(c) H has a right length function

(d) H has BF

(e) $\bigcap_{n \geq 0} (H \setminus H^\times)^n = \emptyset$

If these conditions are satisfied then H is unit-cancellative.

Proof: (a) $\Rightarrow$ (b) $\Rightarrow$ (c) ✓

(c) $\Rightarrow$ (d) Let $\lambda: H \to \mathbb{Z}_{\geq 0}$ be right length function

$\Longrightarrow$ $\max L(a) \leq \lambda(a)$ for all $a \in H$ (Lemma 8.3(1)) if $L(a) \neq \emptyset$.

H is atomic: Induction on $\lambda(a)$.

$\lambda(a) = 0$: $a \in H^\times$ (Lemma 8.3(1)) ✓

$\lambda(a) > 0$: W.r.t. a is not an atom.

So $a = b_1 c_1 \quad b_1, c_1 \in H \setminus H^\times, \quad \lambda(b_1) < \lambda(a)$

$\xRightarrow{IH}$ b_1 is product of atoms.

If c_1 is an atom, we are done. Otherwise, $c_1 = b_2 c_2, \quad b_2, c_2 \in H \setminus H^\times$,

$a = b_1 b_2 c_2 \quad , \quad \lambda(b_1 b_2) < \lambda(a) \longrightarrow b_1 b_2$ is product of atoms

Continuing, $\quad a = \underbrace{b_1 \cdots b_k}_{\text{product of atoms}} \underbrace{c_k}_{\in H \setminus H^\times}$

$\lambda(a) > \lambda(b_1 \cdots b_k) \geq k \Longrightarrow$ process stops after $\leq \lambda(a) - 1$ steps.

$\Longrightarrow$ eventually c_k is an atom.

(d) $\rightarrow$ (e) Suppose $\bigcap_{n \geq 0} (H \setminus H^\times)^n \neq \emptyset$. Let $a \in \bigcap_{n \geq 0} (H \setminus H^\times)^n$

$\Longrightarrow \forall n \geq 0 \; \exists a_1, \ldots, a_n \in H \setminus H^\times : \quad a = a_1 \cdots a_n$

$\xRightarrow{H \text{ atomic}}$ each a_i is a product of atoms

$\Longrightarrow \sup L(a) \geq n$.

(e) $\Rightarrow$ (a) Set $\lambda(a) := \max\{n \geq 0 : \; a \in (H \setminus H^\times)^n\}$. $\square$

<u>Lemma 8.5</u> Let H, D be monoids.

(1) Suppose $\varphi : H \to D$ is a monoid hom. with $\varphi^{-1}(D^\times) = H^\times$
If D has BF, then so does H

(2) If $H \subseteq D$ s.t. $H \cap D^\times = H^\times$, and D has BF, then H has BF

(3) If H has BF, so do $H^\bullet$ and $Z(H)$,

(4) Restricted products of BF-monoids are BF.

(5) Let $H = \bigcap_{i \in I} H_i$ with $H \subseteq H_i \subseteq D$ be an intersection of finite type. If each H_i has BF, so does H

Proof: (1) Let $\lambda : D \to \mathbb{Z}_{\geq 0}$ be a right length function
Let $a, b, c \in H$, $c \notin H^\times$, $a = bc$

$\Rightarrow \varphi(a) = \varphi(b)\varphi(c)$ and $\varphi(c) \notin D^\times \Rightarrow \lambda(\varphi(a)) > \lambda(\varphi(b))$

$\rightarrow \lambda \circ \varphi$ is a right length fn. for H

(2), (3): Apply (1) with $H^\circ \hookrightarrow H$, $\mathcal{Z}(H) \to H$

(4) Let $H := \prod_{i \in J} H_i$, H_i BF. Let $a = (\alpha_i)_{i \in I} \in H$ non-unit,

$I' := \{ i \in I : \alpha_i \notin H_i^\times \}$. $\longrightarrow$ $|I'| < \infty$.

Suppose $a = a_1 \cdots a_k \in H$, $a_j \in H \setminus H^\times$, $a_j = (\alpha_{j,i})_{i \in J}$

$\forall j \in [1, k]\ \exists i \in I : \alpha_{j,i} \notin H_i^\times$. H_i unit-cancellative $\Rightarrow i \in I'$.

$\Rightarrow k \leq \sum_{i \in I'} \max L_{H_i}(\alpha_i)$.

(5) Let $a \in H$. If $\forall i : a \in H_i^\times \Rightarrow a^{-1} \in H \rightarrow a \in H^\times$

$\Rightarrow H \to \prod_{i \in J}^{c} H_i$. + (1) give the claim. $\square$

The submonoid H° ⚠ Factorizations of $a \in H^\circ$ with H or H° may differ, because $H^\circ \subseteq H$ need not be divisor-closed.

Let $S \subseteq H$ be monoids

.) $S \subseteq H$ is right saturated if $\forall a, b \in S,\ c \in H : a = bc \Rightarrow c \in S$

.) —"— divisor closed if $\forall a \in S\ \forall b, c \in H : a = bc \Rightarrow b, c \in S$

.) —"— right Ore set if $\forall a \in H\ \forall b \in S : aS \cap bH \neq \emptyset$

Lemma 8.6 (1) Let H be a monoid. If H is cancellative or H° is a right Ore set, then H° is right saturated in H.

(2) If R is a domain or a ~~prime Goldie~~ semiprime right Goldie ring, then $R^\circ \subseteq R$ is divisor closed

Proof. (1) H cancellative $\Rightarrow H = H^{\circ}$ ✓

Suppose H° is right Ore, $a, b \in H^{\circ}$, $c \in H$: $a = bc$.

(i) $x, y \in H$: $cx = cy \Rightarrow ax = ay \Rightarrow x = y$ ✓

(ii) $x, y \in H$: $xc = yc$. Let $b' \in H$, $a' \in H^{\circ}$: $ab' = ba'$

$\Rightarrow ba' = ab' = bcb' \overset{b \in H^{\circ}}{\Longrightarrow} a' = cb'$

$\Rightarrow xa' = xcb' = ycb' = ya' \overset{a' \in H^{\circ}}{\Longrightarrow} x = y.$

(2) R domain ✓. R prime Goldie:

$a \in R^{\circ} \Leftrightarrow aR \leq_e R_R \Leftrightarrow Ra \leq_e R_R$

If $a = bc$, $b, c \in R \Rightarrow aR \subseteq bR$, $Ra \subseteq Rc \Rightarrow b, c \in R^{\circ}$. $\square$

Suppose $H^{\circ} \subseteq H$ is right saturated,

$[aH, H] := \{ bH : b \in H^{\circ}, aH \subseteq bH \subseteq H \}$ poset $(\subseteq)$

·) If $a = u_1 \cdots u_n$, u_i atoms

$\Rightarrow aH \subsetneq u_1 \cdots u_{n-1} H \subsetneq u_1 \cdots u_{n-2} H \subsetneq \cdots \subsetneq u_1 H \subsetneq H$

is a finite max. chain in $[aH, H]$

·) If $aH = a_n H \subsetneq a_{n-1} H \subsetneq \cdots \subsetneq a_1 H \subsetneq 1H$ is a max. chain in $[aH, H]$, let u_i s.t. $u_1 = a_1$, $a_i = a_{i-1} u_i$

$\Rightarrow u_i \in H^{\circ}$, $u_1 \cdots u_n$ is a fact of a.

Bij. (rigid) fact of $a \leftrightarrow$ max. chains in $[aH, H]$

Then:

H° atomic $\Leftrightarrow [aH, H]$ contains a finite max. chain.

H° has BF $\Leftrightarrow \exists$ uniform bound on chains in $[aH, H]$

H° is half-factorial $\Leftrightarrow$ each $[aH, H]$ has finite max. chain & these chain all have the same length

<u>Lemma 8.7</u> Let H be a monoid s.t. $H^\times \subseteq H$ is ~~.~~ ! saturated. If H satisfies the ACC on principal left ideals & right ideals, then $H^\bullet$ is atomic.

<u>Proof:</u> Ascending chains $Ha_1 \subsetneq Ha_2 \subsetneq \dots$ in $[Ha, H]$ correspond to desc. chains

$b_1 H \supsetneq b_2 H \supsetneq \dots$ in $[aH, H]$

by picking $a = b_i a_i$ $\Big(a_i = \overset{\in H^\bullet}{d_i}\, a_{i+1} \Rightarrow a = b_i a_i = b_{i+1} a_{i+1} = b_i d_i a_{i+1} \Rightarrow b_i d_i = b_{i+1} \Big)$

$\Rightarrow [aH, H]$ has ACC & DCC $\Rightarrow$ every chain can be refined to a maximal one. $\square$

8.2 BF rings

<u>Def 8.8</u> A ring R has bounded factorizations (BF) if $R^\bullet$ is a BF-monoid.

<u>Thm 8.9</u> Let R be a commutative noetherian domain. Then R has BF.

<u>Proof:</u> <u>Recall:</u> (1) If $I \trianglelefteq R$, then $\bigcap_{n \geq 0} I^n = 0$ (Krull's Intersection Theorem)

(2) Every $a \in R \setminus \{0\}$ is contained in at most finitely many height-1 prime ideals. If $a \notin R^\times$ it is contained in at least one.

Set
$$\lambda(a) := \sum_{\substack{\mathfrak{p} \trianglelefteq R \\ \mathfrak{p} \text{ height-1 prime ideal}}} \max\{n \geq 0 : a \in \mathfrak{p}^n\}. \qquad (a \in R \setminus \{0\})$$

$\overset{(1),(2)}{\Longrightarrow} \lambda(a) < \infty.$

Let $a = bc$, $c \notin R^\times$,
let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be the primes minimal over b, let, (1.30) $* n_i = \max\{\_\}$)
$\mathfrak{q}$ be a prime containing c.

<u>Case 1</u>: $\mathfrak{q} \notin \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\} \Rightarrow a \in \mathfrak{q}$, $a \in \mathfrak{p}_i^{n_i}$

$\Rightarrow \lambda(a) > \lambda(b)$

<u>Case 2</u>: $\mathfrak{q} = \mathfrak{p}_1$ (wrl.) $\Rightarrow a \in \mathfrak{p}_1^{n_1+1} \cap \mathfrak{p}_2^{n_2} \cap \dots \cap \mathfrak{p}_t^{n_t}$

$\Rightarrow \lambda(a) > \lambda(b)$. □

We shall prove:

<u>Thm 8.10</u>: Let R be a prime FBN ring s.t. every nonzero ideal contains a nonzero central element. Then R is a BF-ring.

Suppose R is a semiprime right Goldie and $P_1, \dots, P_n$ are the minimal prime ideals, $Q \cong Q_1 \times \dots \times Q_n$.
For an R-module M, define

$$V_{P_i}(M) := \operatorname{length}\left(M/MP_i \otimes_R Q_i\right)$$
$$:= \rho_{R/P_i}\left(M/MP_i\right)$$

$M/MP_i \cong M \otimes R/P_i$, so $V_{P_i}(M)$ counts the multiplicity of the unique simple Q_i-module in $M \otimes_R Q$.

(So: $\rho_R(M) = V_{P_1}(M) + \dots + V_{P_n}(M)$.)

<u>Lemma 8.11</u> R semiprime right Goldie, $P \triangleleft R$ min. prime.

(1) $V_P : \text{Mod-}R \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is additive on SES

(2) $V_P(M) = 0 \Leftrightarrow M/MP$ is R/P-torsion.

<u>Proof</u>: Lemmas 6.8, 6.9. □

<u>Lemma 8.12</u> Let R be a right noetherian ring with prime radical N, P a minimal prime of R, M R-module. If $M=M_0\supseteq M_1\supseteq \dots \supseteq M_m=0$, $M=M_0'\supseteq M_1'\supseteq\dots\supseteq M_n'=0$ are submodule series s.t. $M_iN\subseteq M_{i+1}$, $M_i'N\subseteq M_{i+1}$, $(\forall i)$

Then

$$\sum_{i=1}^{m} v_{P/N}\left(M_i/M_{i+1}\right)=\sum_{i=1}^{n} v_{P/N}\left(M_i'/M_{i+1}'\right)$$

<u>Proof</u>: Schreier refinement thm (cf. Prop 6.6)

<u>Def 8.13</u> Let R be a right noeth. ring, P minimal prime, N prime radical. For M_R an R-module use L.8.12 to define $v_P(M)$.

<u>Note</u>: If M_R R-module, $B\lhd R$, $B\subseteq \operatorname{ann}(M)$, and $P\lhd R$ is a prime ideal minimal over B, then $v_{P/B}(M)$ makes sense, as M is a R/B-module

<u>Lemma 8.14</u> Let R be a right noetherian ring, M_R R-module. Let $A:=\operatorname{ann}(M_R)$, $B\lhd R$. Suppose $P\lhd R$ is a prime ideal s.t. $B\subseteq A\subseteq P$ and P is minimal over B (and A),

Then

$$\underset{(\text{in } R/A)}{v_{P/A}(M)}=\underset{(\text{in } R/B)}{v_{P/B}(M)}$$

<u>Proof</u>: Let N be the prime radical of B. R/B right noeth
$\Longrightarrow \exists n\geq 1:\ N^n\subseteq B\subseteq A$. Consider

$$M\supseteq MN\supseteq MN^2\supseteq\dots\supseteq MN^n=0.$$

Each factor $L_i:=MN^{i-1}/MN^i$ is annihilated by A (bec. M is)

$$\Longrightarrow L_i \otimes_R R/P \cong L_i \otimes_{R/A} (R/A)/(P/A) \cong L_i \otimes_{R/B} (R/B)/(P/B))$$

$\Longrightarrow v_{P/B}(L_i) = v_{P/A}(L_i)$ (meaningful: P is min. /A or /B)

$\Longrightarrow v_{P/A}(M) = v_{P/B}(M)$. $\square$

Lemma 8.15 Let R be a FBN ring with prime radical N, let $c \in \mathcal{C}(N)$. Then cR contains a nonzero ideal of R

(W/o proof: Lemma 2.1 in Chatters, Gilchrist, "The Principal Ideal Theorem in Prime Noetherian Rings", Glasgow Math J, 1986)

Uses Krull-dim + weak ideal invariance.

Prop 8.16 Let R be a prime FBN ring s.t. every nonzero ideal contains a nonzero central element. Let $a \in R^\circ \setminus R^\times$. Then there exists a Krull-1-prime P with $\operatorname{ann}(R/aR) \subseteq P$

Proof: $A := \operatorname{ann}(R/aR)$ is the max. two-sided ideal in aR.

R bounded $\xrightarrow{aR \text{ essential}}$ $0 \neq A \subsetneq R$.

N – prime radical of A

$\xrightarrow{8.15}$ $c \notin \mathcal{C}(N) \Longrightarrow \exists$ minimal prime P over A s.t. $a \notin \mathcal{C}(P)$

$\xrightarrow{P.IT}$ $ht(P) = 1$ $\square$

<u>Lemma 8.17</u> R bounded semiprim right Goldie ring
If $M_1, \dots, M_n$ are f.g. torsion modules, $\exists\, 0 \neq I \triangleleft R$ s.t.
$M_i I = 0 \ \forall i$

<u>Proof:</u> $M_i = \langle m_{i,1}, \dots, m_{i,k} \rangle_R$, Pick $x_{ij} \in R'$ s.t.

$m_{11} x_{11} = 0$, $(m_{12} x_{11}) x_{12} = 0$, $(m_{13} x_{11} x_{12}) x_{13} = 0$, ...

$(m_{21} x_{11} \cdots x_{1k}) x_{21} = 0$, ...

$\vdots$

$x = x_{11} \cdots x_{nk} \in R^\circ \implies m_{ij} x = 0$ for <u>all</u> $i \in [1,n]$, $j \in [1,k]$.

R semiprim right Goldie $\implies xR \leq_e R_R \xrightarrow{R \text{ bdd.}} \exists$ essential $I \triangleleft R$ s.t. $I \subseteq xR$

If $m = \sum m_{ij} r_j \in M_i \implies mI \subseteq \sum m_{ij} r_j I \subseteq \sum m_{ij} I = 0$.

$\implies I \subseteq \operatorname{ann}(M_i)$. $\square$

<u>Lemma 8.18</u> R FBN ring. If $a \in R^\circ \setminus R^\times$ and P is a prim ideal minimal over $A = \operatorname{ann}(R/aR)$, then $\nu_{P/A}(R/aR) > 0$.

<u>Proof.</u> N prime radical of A, $P_1 = P, P_2, \dots, P_k$ min primes $/A$
(pw dis)

$Y := P_2 \cap \dots \cap P_k \cap R$.

Suppose $\nu_{P/A}(R/aR) = 0$.

$R/aR = M_0 \supseteq M_1 \supseteq \dots \supseteq M_n = 0$ with $M_{i-1} N \subseteq M_i$.

$\implies \nu_{P/A}(M_{i-1}/M_i) = 0 \ \forall i \implies M_{i-1}/(M_{i-1}P + M_i)$ is R/P torsion

$\overset{8.17}{\implies} \exists$ ideal $X \supsetneq P$ annihilating all $M_{i-1}/(M_{i-1}P+M_i)$

Y annihilates $(M_{i-1}P + M_i)/M_i$

$\implies XY$ annihilates $M_{i-1}/M_i \implies (XY)^n \subseteq A \subseteq N$

$\implies XP_2 \cdots P_n \subseteq N \subseteq P_1 \implies X \subseteq P_1$ ↯. $\square$

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<u>Proof of Thm 8.10</u> Let $\mathfrak{X}(R)$ be the set of height-1-primes of R. For $a \in R^\circ$, $A = \operatorname{ann}(R/aR)$,

$$\lambda(a) := \sum_{\substack{P \in \mathfrak{X}(R) \\ A \subseteq P}}{}' v_{P/A}(R/aR)$$ is a superadditive length function

<u>Claim:</u> λ is superadditive length fn.

(i) Let $a \in R^\circ \setminus R^\times$. $A = \operatorname{ann}(R/aR) \neq 0$, because R is bounded and $aR \leq_e R_R$.

$\xRightarrow{\text{Prop 8.16}}$ $\exists$ height-1-prime $P \supseteq A$, P must be minimal $/A$ $(A \neq 0)$

$\xRightarrow{\text{L.8.18}}$ $v_{P/A}(R/aR) > 0 \implies \lambda(a) > 0$.

(ii) Let $a = bc$, $b, c, a \in R^\circ$; $B = \operatorname{ann}(R/bR)$, $C = \operatorname{ann}(R/cR)$

$$\begin{array}{l} R \\ | \\ bR \\ | \\ aR \end{array} \quad R/bR,\quad bR/bcR \cong R/cR$$

Since R/bR, R/cR appear as submodule/factor of R/aR, $A \subseteq B, C$

$$\lambda(a) = \sum_{\substack{P \in \mathfrak{X} \\ A \subseteq P}}{}' v_{P/A}(R/aR) = \sum_{\substack{P \in \mathfrak{X} \\ A \subseteq P}}{}' \left(v_{P/A}(R/bR) + v_{P/A}(R/cR) \right)$$

$$\geq \sum_{\substack{P \in \mathfrak{X} \\ B \subseteq P}}{}' v_{P/A}(R/bR) + \sum_{\substack{P \in \mathfrak{X} \\ C \subseteq P}}{}' v_{P/A}(R/cR)$$

$$\overset{\text{L.8.14}}{=} \sum_{\substack{P \in \mathfrak{X} \\ B \subseteq P}}{}' v_{P/B}(R/bR) + \sum_{\substack{P \in \mathfrak{X} \\ C \subseteq P}}{}' v_{P/C}(R/cR) = \lambda(b) + \lambda(c) \qquad \square$$

<u>Cor 8.19</u> Every noetherian prime PI ring is BF.