Noncommutative Noetherian Rings

Daniel Smertnig

- Module Theory: Tensor Products
- PI Rings

Graz, Sommersemester 2017

21 Einheiten zu 60 Minuten

I Some Module Theory: Tensor Products

0. Background

All rings will be associative and unital, unless stated otherwise.

Def. Let R be a ring.

(1) A (left) **R-module** is an abelian group $(M,+)$ together with a map $\cdot : R\times M \longrightarrow M$, $(r,m)\longmapsto r\cdot m = rm$ s.t. $\forall r,s\in R\ \forall m,n\in M$:

- $r(m+n) = rm+rn$
- $(r+s)m = rm+sm$
- $(rs)m = r(sm)$
- $1m = m$

(2) Let ${}_RM, {}_RN$ be modules (as indicated). A **homomorphism** is a map $f: M\longrightarrow N$ s.t. $\forall r\in R\ \forall m,n\in M$:

- $f(m+n) = f(m)+f(n)$
- $f(rm) = r f(m)$.

- Right modules are defined analogously.
- R-Mod denotes the category of left R-modules / Mod-R right R-modules. ($\mathrm{Hom}_R(M,N)$ denotes the set of homomorphisms from M to N.)
- A left R-module structure is equivalently given by a ring hom. $R\longrightarrow \mathrm{End}_{\mathbb{Z}}(M)$.

["$\mapsto$": $r\longmapsto (m\longmapsto rm)$

"$\leftarrow$": Let $\varphi: R\to \mathrm{End}_{\mathbb{Z}}(M)$. $r\cdot m := \varphi(r)(m)$]

— right ———

$R^{op}\longrightarrow \mathrm{End}_{\mathbb{Z}}(M)$.

- A left R-module structure is the same thing as a right R^{op}-module structure!
- If R is commutative, $R=R^{op}$, and left/right modules are the same thing!

Def: Let R, S be rings. A (R,S)-bimodule is a left R-module and right S-module ${}_R M_S$ s.t.
$\forall r \in R\ \forall s \in S\ \forall m \in M$: $r(ms) = (rm)s$.

1. Tensor products of modules

Def 1.1 Let R be a ring, M_R a right module, and ${}_R N$ a left module. Let F be the free abelian group with basis $M \times N$, and $K \subseteq F$ the subgroup generated by all elements of the forms

- $(m+m', n) - (m,n) - (m',n)$
- $(m, n+n') - (m,n) - (m,n')$
- $(mr, n) - (m, rn)$

for $r \in R$, $m, m' \in M$, $n, n' \in N$.

$M \otimes N := M \otimes_R N := F/K$ is the tensor product of M and N.
(It is an abelian group)
The coset $(m,n) + K$ is denoted by $m \otimes n$.

(*) An element of $M \otimes_R N$ is called a tensor and can be represented (not uniquely) as
$\sum_{i=1}^{r} a_i (m_i \otimes n_i)$ $\quad r \in \mathbb{N}_0,\ a_i \in \mathbb{Z},\ m_i \in M,\ n_i \in N.$

A tensor of the form $m \otimes n$ is called an elementary tensor, but not every tensor is of such a form!

Exm: Let $R = \mathbb{Z}$, $M = N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ free abelian group of rank 2.
Then $(e_i \otimes e_j \mid i,j \in [1,2])$ is a $\mathbb{Z}$-basis for $M \otimes_{\mathbb{Z}} N$. (Proof later!)

Claim: $e_1 \otimes e_1 + e_2 \otimes e_2$ is not elementary.

Proof: Suppose it were. $e_1 \otimes e_1 + e_2 \otimes e_2 = (a_1 e_1 + a_2 e_2) \otimes (b_1 e_1 + b_2 e_2)$
$= a_1 b_1 (e_1 \otimes e_1) + a_2 b_2 (e_2 \otimes e_2) + a_1 b_2 (e_1 \otimes e_2) + a_2 b_1 (e_2 \otimes e_1)$

$\Longrightarrow a_1 b_2 = 0 \quad \longrightarrow a_1 b_1 = 0$ or $a_2 b_2 = 0$ $\quad \lightning$

•) Let A be an abelian group. A map $f: M\times N \to A$ is middle linear (or (bilinear) R-balanced) if $\forall r\in R\ \forall m,m'\in M\ \forall n,n'\in N$:

$$f(m+m',n)=f(m,n)+f(m',n) \qquad f(m,n+n')=f(m,n)+f(m,n')$$

$$f(mr,n)=f(m,rn).$$

•) The map $i: M\times N \longrightarrow M\otimes_R N$, $(m,n)\longmapsto m\otimes n$ is middle linear.
It is called the canonical middle linear map.
$M\otimes_R N$ is always (implicitly) considered together with the map i!

Careful: i need not be injective, e.g.

$M=N=R=\mathbb{Z}$: $\quad 2\otimes 3 = 6\otimes 1 = 1\otimes 6$

Thm 1.2 (Universal Property) Let R be a ring, M_R, ${}_RN$ modules, A an abelian group. If $f: M\times N\to A$ is middle linear, then there exists a unique group hom. $\bar f: M\otimes_R N\to A$ such that $\bar f\circ i = f$, where $i: M\times N\to M\otimes_R N$, $(m,n)\mapsto m\otimes n$.

$$\begin{tikzcd} M\times N \arrow[r, "f"] \arrow[d, "i"'] \arrow[dr, phantom, "///"] & A \\ M\otimes_R N \arrow[ur, dashed, "\exists!\,\bar f"'] & \end{tikzcd}$$

Proof: Existence. Let $F\supset K$ be as in Def. 1, $\pi: F\longrightarrow F/K$, $x\mapsto x+K$.
Since F is a free abelian group with basis $M\times N$, there exists a unique group hom. $\tilde f: F\longrightarrow A$ with $\tilde f(m,n)=f(m,n)$ for $m\in M$, $n\in N$.

Claim: $K\subseteq \operatorname{Ker}(\tilde f)$

Proof: $\tilde f\big((m+m',n)-(m,n)-(m',n)\big) = \tilde f((m+m',n)) - \tilde f((m,n)) - \tilde f((m',n))$
$\quad = f(m+m',n) - f(m,n) - f(m',n) = 0$

...

$\Longrightarrow \exists\, \bar f: F/K = M\otimes N \longrightarrow A$, $\quad m\otimes n = (m,n)+K \longmapsto \tilde f(m,n) = f(m,n)$ $\square$

$\Longrightarrow \bar f\circ i = f$

<u>Uniqueness:</u> Suppose $h: M\otimes_R N \to A$ is another group hom with

$h\circ i = f = \bar{f}\circ i$

$\Longrightarrow \forall m\in M, n\in N: \quad h(m\otimes n) = \bar{f}(m\otimes n)$

Since $M\otimes N = {}_{\mathbb{Z}}\langle\{m\otimes n \mid m\in M, n\in N\}\rangle$, this implies $h=\bar{f}$. $\square$

- Objects satisfying a UP are unique up to unique isomorphism.

<u>That is:</u> Suppose P is an abelian group with a map $j: M\times N \to P$ such that (P,j) also satisfies the UP:

$$\begin{tikzcd}
M\times N \arrow[r, "f"] \arrow[d, "j"'] \arrow[dr, phantom, "///"] & A \\
P \arrow[ur, dashed, "\exists!\bar{f}"'] &
\end{tikzcd}$$

$\Longrightarrow \exists!\ \varphi: M\otimes_R N \xrightarrow{\sim} P$ such that $j=\varphi\circ i$.

<u>Proof:</u>

$$\begin{tikzcd}
M\times N \arrow[r, "j"] \arrow[d, "i"'] \arrow[dr, phantom, "///"] & P \arrow[dl, dashed, bend left, "\exists!\psi"] \\
M\otimes N \arrow[ur, dashed, "\exists!\varphi"] &
\end{tikzcd}$$

$j=\varphi\circ i$
$i=\psi\circ j$

$$\begin{tikzcd}
M\times N \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, "///"] & M\otimes N \\
M\otimes N \arrow[ur, shift left, "id"] \arrow[ur, shift right, "\psi\circ\varphi"'] &
\end{tikzcd}$$

$id\circ i = i$
$\psi\circ\varphi\circ i = \psi\circ j = i$

$\xRightarrow{\text{uniqueness}} id_{M\otimes N} = \psi\circ\varphi$

Analogously: $id_P = \varphi\circ\psi$

$\square$

- The tensor product can therefore also be defined as an abelian group P together with a map $j: M\times N \to P$ s.t. (P,j) satisfies the UP.

Cor 1.3 Let R be a ring, $M_R, M'_R, {}_RN, {}_RN'$ modules.
If $f: M_R \to M'_R$, $g: {}_RN \to {}_RN'$ are module homomorphisms, then there exists a unique group hom. $f\otimes g: M\otimes N \to M'\otimes N'$ with

$$f\otimes g\,(m\otimes n) = f(m)\otimes g(n).$$

WARNING: The notation $f\otimes g$ introduced here must not be confused with the elementary tensor $f\otimes g \in \mathrm{Hom}(M,M')\otimes_{\mathbb{Z}} \mathrm{Hom}(N,N')$

Proof: • $\varphi: M\times N \to M'\otimes_R N'$, $(m,n) \longmapsto f(m)\otimes g(n)$ is middle linear.

Proof. $\varphi(m+m',n) = f(m+m')\otimes g(n) = [f(m)+f(m')]\otimes g(n) = f(m)\otimes g(n) + f(m')\otimes g(n)$
$= \varphi(m,n) + \varphi(m',n)$

$\varphi(m,n+n') = \dots = \varphi(m,n) + \varphi(m,n')$

$\varphi(mr,n) = f(mr)\otimes n = f(m)r\otimes n = f(m)\otimes rn = \varphi(m,rn)$ ⌋

$\overset{UP}{\Longrightarrow} \exists!\ f\otimes g: M\otimes N \to M'\otimes N',\ f\otimes g(m\otimes n) = \varphi(m,n) = f(m)\otimes g(n).$

The map is unique, since for every such map $h: M\otimes N \to M'\otimes N'$ we have $h\circ i = \varphi$ when $i: M\times N \to M\otimes N$.

□

Prop 1.4 Let R be a ring, $M_R, M'_R, M''_R, {}_RN, {}_RN', {}_RN''$ modules.

1) If $f_1, f_2: M \to M'$, $g_1, g_2: N \to N'$, then

$$f_1\otimes(g_1+g_2) = f_1\otimes g_1 + f_1\otimes g_2, \qquad (f_1+f_2)\otimes g_1 = f_1\otimes g_1 + f_2\otimes g_1$$

2) If $M \xrightarrow{f} M' \xrightarrow{f'} M''$, $N \xrightarrow{g} N' \xrightarrow{g'} N''$, then

$$(f'\otimes g')\circ(f\otimes g) = (f'\circ f)\otimes(g'\circ g)$$

3) $M_R\otimes - : R\text{-Mod} \to \mathbb{Z}\text{-Mod}$ ($=Ab$), $-\otimes_R N: \text{Mod-}R \to \mathbb{Z}\text{-Mod}$ ($=Ab$) are additive functors. $(M\otimes f := id_M\otimes f)$

Proof. 1), 2) easily checked on the elementary tensors.
3) ✓

Remark: Since $M_R\otimes -$, $-\otimes_R N$ are additive, they preserve split exact sequences. Hence they preserve finite direct sums.
(We see later that they preserve arbitrary direct sums.)

Thm 1.5 Let R be a ring.

(1) Let M_R be a module, and ${}_RN' \xrightarrow{f} {}_RN \xrightarrow{g} {}_RN'' \to 0$ an exact sequence of modules. Then

$$M \otimes N' \xrightarrow{M \otimes f} M \otimes N \xrightarrow{M \otimes g} M \otimes N'' \to 0$$

is exact.

That is, $M \otimes_R -$ is right exact.

(2) $- \otimes_R N$ (when ${}_RN$ is a module) is right exact.

Proof: (1) Must show: (i) $\mathrm{im}(M \otimes g) = M \otimes N''$

(ii) $\mathrm{im}(M \otimes f) = \ker(1_M \otimes g)$

(i) Since $M \otimes N'' = {}_{\mathbb{Z}}\langle \{ m \otimes n \mid m \in M, n \in N''\}\rangle$ suffices to show $\forall m \in M \forall n \in N''$ $m \otimes n \in \mathrm{im}(M \otimes g)$

g epi $\Rightarrow \exists n_0 \in N : \quad g(n_0) = n$

$\Rightarrow \quad m \otimes n = m \otimes g(n_0) = (M \otimes g)(m \otimes n_0)$

(ii) „$\subseteq$": $(M \otimes f) \circ (M \otimes g) = M \otimes (g \circ f) = M \otimes 0 \overset{\text{because } a \otimes 0 = 0 \ \forall a \in M}{=} 0$

„$\supseteq$": Since we have shown „$\subseteq$", $M \otimes g$ induces an epimorphism

$$\overline{M \otimes g} : \frac{M \otimes_R N}{\mathrm{im}(M \otimes f)} \longrightarrow M \otimes N''.$$

If we show that this is an isomorphism, then $\mathrm{im}(M \otimes f) = \ker(M \otimes g)$ follows.

Let $\pi : M \otimes N \to M \otimes N / \mathrm{im}(M \otimes f)$ be the canonical epi.

Let $m \in M$, $n \in N'' \overset{g \text{ epi}}{\Longrightarrow} \exists n_0 \in N : \quad g(n_0) = n''$

Claim: If $n_0, n_1 \in N''$: $g(n_0) = g(n_1) = n''$, then $\pi(m \otimes n_0) = \pi(m \otimes n_1)$

Proof: $g(n_1 - n_0) = 0 \overset{\mathrm{im} f = \ker g}{\Longrightarrow} \exists x \in N' : \quad f(x) = n_1 - n_0$

$\pi(m \otimes n_0) = \pi(m \otimes n_0 + m \otimes f(x)) = \pi(m \otimes (n_0 + f(x))) = \pi(m \otimes n_1)$ $\square$(Claim)

Define $h_0 : \begin{cases} M \times N'' \longrightarrow M \otimes_R N / \mathrm{im}(M \otimes f) \\ (m, n) \longmapsto \pi(m \otimes n_0) \quad \text{with } n_0 \in g^{-1}(\{n\}). \end{cases}$

h_0 is middle linear [✓]

$\Rightarrow \exists h : M \otimes N'' \longrightarrow M \otimes N / \mathrm{im}(M \otimes f), \quad m \otimes n \longmapsto \pi(m \otimes n_0)$ if $g(n_0) = n$.

Let $m \in M$, $n \in N''$, $n_0 \in N$ s.t. $g(n_0) = n$

$\overline{M \otimes g} \circ h\,(m \otimes n) = \overline{M \otimes g}\,(\pi(m \otimes n_0)) = M \otimes g\,(m \otimes n_0) = m \otimes n.$

Let $m \in M$, $n \in N$:

$h \circ \overline{M \otimes g}\,(\pi(m \otimes n)) = h \circ (M \otimes g)(m \otimes n) = h(m \otimes g(n)) = \pi(m \otimes n)$.

(2) similar. / opposite rings: $M_R \otimes_R {}_R N = N_{R^{op}} \otimes_{R^{op}} {}_{R^{op}}M$ $\square$

Corollary 1.6 Let R be a ring, $M'_R \xrightarrow{i} M_R \xrightarrow{f} M''_R \to 0$,

${}_RN' \xrightarrow{j} {}_RN \xrightarrow{g} {}_RN'' \to 0$ exact.

Then $f \otimes g : M \otimes N \to M'' \otimes N''$ is an epi and

$$\ker(f \otimes g) = \operatorname{im}(i \otimes N) + \operatorname{im}(M \otimes j)$$

Proof: $f \otimes g : M \otimes N \xrightarrow{M \otimes g} M \otimes N'' \xrightarrow{f \otimes N''} M'' \otimes N''$

$f \otimes g = (f \otimes N'') \circ (M \otimes g) \overset{\text{Thm 1.5}}{\Longrightarrow} f \otimes g$ epi

„$\supseteq$": $(f \otimes g) \circ (i \otimes N) = (f \circ i) \otimes g = 0 \otimes g = 0$

$(f \otimes g) \circ (M \otimes j) = f \otimes (g \circ j) = f \otimes 0 = 0$

„$\subseteq$" $f \otimes g : \quad M \otimes N \xrightarrow{M \otimes g} M \otimes N'' \xrightarrow{f \otimes N''} M'' \otimes N''$

$\ker(f \otimes N'') \overset{\text{Thm 1.5}}{=} \operatorname{im}(i \otimes N'') = i \otimes N''(M' \otimes N'') = i \otimes N''(M' \otimes g(N)) = (i \otimes N'') \circ (M' \otimes g)(M' \otimes N)$

$= i \otimes g(M' \otimes N)$

Let $x \in \ker(f \otimes g)$

$\Rightarrow M \otimes g(x) \in \ker(f \otimes N'') = \operatorname{im}(i \otimes g)$

$\Rightarrow \exists y \in M' \otimes N : \quad M \otimes g(x) = i \otimes g(y)$

$z := x - i \otimes N(y)$

$\Rightarrow M \otimes g(z) = 0 \Rightarrow z \in \ker(M \otimes g) = \operatorname{im}(M \otimes j)$

$\Rightarrow x \in \operatorname{im}(i \otimes N) + \operatorname{im}(M \otimes j)$.

$\square$

Example: $K_R \leq M_R$, ${}_RL \leq {}_RN$

$$\Rightarrow M/K \otimes_R N/L \cong M \otimes_R N \Big/ {}_{\mathbb{Z}}\langle \{k \otimes n \mid k \in K, n \in N\} \cup \{m \otimes \ell \mid m \in M, \ell \in L\} \rangle$$

„$K \otimes N + M \otimes L$" (image in $M \otimes N$)

Example: i) Let $n, k \in \mathbb{N}$. Then $\mathbb{Z}/n^k\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$.

$$\left\{\begin{array}{l} \mathbb{Z}/n^k\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \\ (a+n^k\mathbb{Z},\ b+n\mathbb{Z}) \longmapsto ab+n\mathbb{Z} \end{array}\right. \quad \text{is middle linear } /\mathbb{Z} \qquad (a, b \in \mathbb{Z})$$

$\Rightarrow \exists$ hom. $\varphi: \mathbb{Z}/n^k\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$

$$(a+n^k\mathbb{Z}) \otimes (b+n\mathbb{Z}) \longmapsto ab+n\mathbb{Z}$$

Let $\psi: \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n^k\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$

$$a+n\mathbb{Z} \longmapsto (1+n^k\mathbb{Z}) \otimes (a+n\mathbb{Z}).$$

Claim: (i) $\varphi \circ \psi = 1_{\mathbb{Z}/n\mathbb{Z}}$, (ii) $\psi \circ \varphi = 1_{\mathbb{Z}/n^k\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}}$

(i) ✓

(ii) $\psi \circ \varphi\big((a+n^k\mathbb{Z}) \otimes (b+n\mathbb{Z})\big) = \psi(ab+n\mathbb{Z}) = (1+n^k\mathbb{Z}) \otimes (\underset{\in \mathbb{Z}}{a}b+n\mathbb{Z}) = (a+n^k\mathbb{Z}) \otimes (b+n\mathbb{Z})$.

ii)
$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n^2\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \to 0$$

$$a+n^2\mathbb{Z} \longmapsto a+n\mathbb{Z}$$

$$b+n\mathbb{Z} \longmapsto nb+n^2\mathbb{Z}$$

$$(a+n\mathbb{Z}, b+n\mathbb{Z}) \longmapsto (na+n^2\mathbb{Z}, b+n\mathbb{Z})$$

$$(a+n^2\mathbb{Z}, b+n\mathbb{Z}) \longmapsto (a+n\mathbb{Z}, b+n\mathbb{Z})$$

$$\begin{array}{ccccccc}
\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \mathbb{Z}/n^2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} & \to & 0 \\
\| \wr & & \| \wr & & \| \wr & & \\
\mathbb{Z}/n\mathbb{Z} & \xrightarrow{0} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0
\end{array}$$

$$a+n\mathbb{Z} \longrightarrow a+n\mathbb{Z}$$

$$a+n\mathbb{Z} \longmapsto 0+n\mathbb{Z}$$

Prop 1.7 Let R be a ring, I, J index sets, $(M_i)_R$, $_R(N_j)$ for $i\in I$, $j \in J$ modules. Then
$$\bigoplus_{\substack{i\in I\\ j\in J}} M_i \otimes_R N_j \cong \Big(\bigoplus_{i\in I} M_i\Big)\otimes_R \Big(\bigoplus_{j\in J} N_j\Big)$$

Proof: $M_R := \bigoplus_{i\in I} M_i$, $\quad {}_RN := \bigoplus_{j\in J} N_j$ [We use the „explicit" construction of a direct sum to simplify notation]

Note that:
$$h: \begin{cases} \Big(\bigoplus_{i\in I} M_i\Big)\times\Big(\bigoplus_{j\in J} N_j\Big) \longrightarrow \bigoplus_{\substack{i\in I\\ j\in J}} M_i\otimes N_j \\ \Big(\sum'_{i\in I} m_i, \sum'_{j\in J} n_j\Big) \longmapsto \sum'_{\substack{i\in I\\ j\in J}} m_i\otimes n_j \end{cases}, \quad m_i\in M_i,\ n_j\in N_j$$
is a middle linear map.

By the UP (Thm 1.2), it suffices to show that $\Big(\bigoplus_{\substack{i\in I\\ j\in J}} M_i\otimes M_j, h\Big)$ satisfies the UP of the tensor product of $M\otimes N$.

Let A be an abelian group, $f: M\times N\to A$ middle linear.

To show: $\exists!^{\text{grp hom}}\ \bar f: \bigoplus_{\substack{i\in I\\ j\in J}} M_i\otimes N_j \longrightarrow A$ s.t. $\bar f\circ h = f$.

Existence: $\tilde f_{i,j}: M_i\times N_j \longrightarrow A$, $(m_i,n_j)\longmapsto f(\overset{\in M}{m_i}, \overset{\in N}{n_j})$
is middle linear.

$\Longrightarrow \exists f_{i,j}: M_i\otimes N_j \to A$, $\quad m_i\otimes n_j \longmapsto f(m_i,n_j)$

Since $\bigoplus_{\substack{i\in I\\ j\in J}} M_i\otimes N_j$ is a direct sum, $\exists$ group hom:
$$\bar f: \bigoplus_{\substack{i\in I\\ j\in J}} M_i\otimes N_j \longrightarrow A \quad \text{s.t.} \quad \bar f|_{M_i\otimes N_j} = f_{ij}$$

Thus: $\forall m_i\in M_i \forall n_j\in N_j$:
$$\bar f\Big(\sum'_{\substack{i\in I\\ j\in J}} m_i\otimes n_j\Big) = \sum'_{\substack{i\in I\\ j\in J}} f_{ij}(m_i\otimes n_j) = \sum'_{\substack{i\in I\\ j\in J}} f(m_i,n_j) = f\Big(\sum'_{i\in I} m_i, \sum'_{j\in J} n_j\Big)$$
$\Longrightarrow \bar f\circ h = f$.

Uniqueness: Necessarily $\bar f(m_i\otimes n_j) = f(m_i,n_j)$ and the $m_i\otimes n_j$ generate $\bigoplus\bigoplus M_i\otimes N_j$ ...

Prop 1.8: Let R, S be rings, ${}_SM_R$, ${}_RN_T$ bimodules.

(1) $M \otimes_R N$ is a (S,T)-bimodule such that
$$s(m\otimes n)t = (sm)\otimes(nt) \quad \text{for } m \in M, n \in N, s \in S, t \in T.$$

mnemonic: ${}_SM\otimes_R N_T = {}_S(M\otimes_R N)_T$

(2) If $f: {}_SM_R \to {}_SM'_R$, $g: {}_RN_T \to {}_RN'_T$ are bimodule hom,
then $f\otimes g: M\otimes_R N \to M'\otimes_R N'$ is a (S,T)-bimodule hom.

Proof: (1) S-left module structure:

Recall that an S-left module structure can be given in terms of a ring hom $S \to \mathrm{End}(M\otimes_R N)$.

For $s \in S$, define $\tilde{\mu}_s: M \times N \to M\otimes_R N$, $(m,n) \to sm\otimes n$.
$\tilde{\mu}_s$ is middle linear [V] Hence $\exists$ group hom $\mu_s: M\otimes_R N \to M\otimes_R N$ s.t. $\mu_s(m\otimes n) = sm\otimes n$ $\forall m \in M, n \in N$.

Claim: $\mu: S \to \mathrm{End}(M\otimes_R N)$, $s \mapsto \mu_s$ is a ring hom.

Proof: $\mu(1_S) = \mu_{1_S} = id_{M\otimes N}$ ✓

$$\mu(s+s')(m\otimes n) = (s+s')m\otimes n = (sm+s'm)\otimes n = sm\otimes n + s'm\otimes n$$
$$= \mu(s)(m\otimes n) + \mu(s')(m\otimes n)$$

$$\mu(ss')(m\otimes n) = ss'm\otimes n = s(s'm)\otimes n = \mu(s)((s'm)\otimes n) = \mu(s)(\mu(s')(m\otimes n))$$
$$= \mu(s)\circ\mu(s')(m\otimes n)$$

$\to \mu(ss') = \mu(s)\circ\mu(s')$.

$\square$ (Claim)

T-right module structure: Analogously construct ring hom $T^{op} \to \mathrm{End}(M\otimes_R N)$

(S,T)-bimodule: Suffices to check on a generating set

$$(s(m\otimes n))t = (sm\otimes n)t = sm\otimes nt = (sm\otimes n)t = (s(m\otimes n))t \quad ✓$$

(2) Need to show: $\forall s \in S\, \forall t \in T\, \forall m \in M, n \in N$: $f\otimes g(s(m\otimes n)t) = s(f\otimes g(m\otimes n))t$

$$f\otimes g(s(m\otimes n)t) = f\otimes g(sm\otimes nt) = f(sm)\otimes g(nt) = sf(m)\otimes g(n)t =$$
$$= s(f(m)\otimes g(n))t$$

Remark: ·) ${}_R M$ is a ${}_R M_{\mathbb{Z}}$-bimodule, ${}_R N$ is a ${}_R N_{\mathbb{Z}}$-bimodule, so this includes the case where e.g. ${}_S M_R$ is a bimodule and ${}_R N$ is a module.

·) All our results (e.g. UP) have natural analogues for bimodules.

·) If ${}_S M_R$ is a bimodule, ${}_S M_R \otimes - : R\text{-Mod} \to S\text{-Mod}$

— ${}_R N_T$ — " —, $- \otimes_R N_T : \text{Mod-}R \to \text{Mod-}T$

are right exact functors.

Special case: R commutative ring (Every R-module is then a bimodule)

·) ${}_R M, {}_R N, {}_R P$ R-modules. A map $f: M \times N \to P$ is R-bilinear if: $\forall m, m' \in M, n, n' \in N, r \in R$:

$$f(m+m', n) = f(m,n) + f(m',n), \qquad f(m, n+n') = f(m,n) + f(m,n')$$

$$r f(m,n) = f(rm, n) = f(m, rn)$$

·) The canonical middle linear map $i: M \times N \to M \otimes_R N$ is bilinear

·) (UP) Let ${}_R M, {}_R N$ be modules. For every bilinear map $f: M \times N \to P$ (where P is an R-module), there exists a unique R-module homomorphism $\bar{f}: M \otimes_R N \to P$ s.t. $\bar{f} \circ i = f$.

[Proof idea: Apply Thm 1.2 (UP), check that $\bar{f}$ is an R-module hom.]

Prop 1.9 Let R, S be rings.

(1) Let M_R, ${}_RN$ be modules

(i) $M \otimes_R N \cong N \otimes_{R^{op}} M$

(ii) $M \otimes_R R_R \cong M_R$ and ${}_RR \otimes_R N \cong {}_RN$

(2) Let M_R, ${}_RN_S$, ${}_SP$ be modules. Then

$$(M \otimes_R N) \otimes_S P \cong M \otimes_R (N \otimes_S P).$$

Proof (Sketch)

(1) i) $\exists$ hom. $M \otimes_R N \to N \otimes_{R^{op}} M$, $m \otimes n \mapsto n \otimes m$,

$N \otimes_{R^{op}} M \to M \otimes_R N$, $n \otimes m \mapsto m \otimes n$.

ii) $\exists$ R-module hom $\begin{cases} M \otimes_R R_R \longrightarrow M_R \\ m \otimes r \longmapsto mr \end{cases}$ $\quad \begin{cases} M_R \longrightarrow M \otimes R_R \\ m \longmapsto m \otimes 1 \end{cases}$

and they are inverse to each other. Similar for ${}_RR \otimes_R N$.

(2) $(M \otimes_R N) \otimes_S P = {}_{\mathbb{Z}}\langle \{ (m \otimes n) \otimes p \mid m \in M, n \in N, p \in P \} \rangle$

$M \otimes_R (N \otimes_S P) = {}_{\mathbb{Z}}\langle \{ m \otimes (n \otimes p) \mid m \in M, n \in N, p \in P \} \rangle$

$\overset{UP}{\Longrightarrow}$ $\forall p \in P\ \exists$ group hom $\varphi_p : \begin{cases} M \otimes_R N \longrightarrow M \otimes_R (N \otimes_S P) \\ m \otimes n \longmapsto m \otimes (n \otimes p) \end{cases}$

$\overset{UP}{\Longrightarrow}$ $\exists$ group hom. $\varphi : \begin{cases} (M \otimes_R N) \otimes_S P \longrightarrow M \otimes_R (N \otimes_S P) \\ (m \otimes n) \otimes p \longmapsto m \otimes (n \otimes p) \end{cases}$

A similar (inverse) hom can be constructed in the other direction.

$\square$

Remark:
- We identify $(M \otimes_R N) \otimes_S P$ and $M \otimes_R (N \otimes_S P)$ and write $M \otimes_R N \otimes_S P$
- Inductively, we define the n-fold tensor product

$$M_1 \otimes_{R_1} M_2 \otimes_{R_2} \ldots \otimes_{R_{n-1}} M_n$$

($R_1, \ldots, R_{n-1}$ rings, $(M_1)_{R_1}$, ${}_{R_{i-1}}(M_i)_{R_i}$, ${}_{R_{n-1}}(M_n)$ (bi)modules)

(Alternatively, this can also be done using n-linear maps)

<u>Recall</u>: If R, S are rings, ${}_R M_S$, N_S (bi) modules, then

$\mathrm{Hom}_S(M, N)$ is a <u>right</u> R-module such that:

$$(fr)(m) = f(rm) \qquad \forall f \in \mathrm{Hom}_S(M,N),\ r \in R,\ m \in M.$$

<u>Theorem 1.10</u> (Adjointness of $\otimes$ and Hom) Let R, S be rings, M_R, ${}_R N_S$, P_S (bi) modules

Then there is a group isomorphism

$$\alpha : \mathrm{Hom}_S(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(N, P))$$

such that

$$(\alpha(f)(m))(n) = f(m \otimes n) \qquad \forall f \in \mathrm{Hom}_S(M \otimes N, P),\ m \in M,\ n \in N$$

<u>Proof (Sketch)</u>:

(i) $\forall f \in \mathrm{Hom}_S(M\otimes N, P)\ \forall m \in M$: $\alpha_{f,m} : N \longrightarrow P$, $n \longmapsto f(m \otimes n)$

is a S-module hom.

(ii) $\forall f \in \mathrm{Hom}_S(M \otimes N, P)$: $M \longmapsto \mathrm{Hom}_S(N,P)$, $m \longmapsto \alpha_{f,m}$ $(n \longmapsto f(m\otimes n))$

is a R-module hom.

$\Longrightarrow$ α is a well-defined function.

(iii) α is a group hom.: $\alpha(f+g) = \alpha(f) + \alpha(g)$

Now, construct an inverse $\beta : \mathrm{Hom}_R(M, \mathrm{Hom}_S(N,P)) \longrightarrow \mathrm{Hom}_S(M \otimes_R N, P)$:

$\forall f \in \mathrm{Hom}_R(M, \mathrm{Hom}(N,P))$: $\quad \beta(f)(m \otimes n) \overset{!}{=} (f(m))(n)$.

(iv) First: $\forall f$ an S-module hom $\beta(f) : (M \otimes_R N)_S \longrightarrow P_S$, $m \otimes n \mapsto f(m)(n)$ exists

(UP of $\otimes$!)

$\Rightarrow$ β well-defined

~~[(v) β is a group homomorphism $(\beta(f+g) = \beta(f) + \beta(g))$]~~

(v) $\beta \circ \alpha = id$, $\alpha \circ \beta = id$.

$\square$

Remark: •) This construction is functorial in M, N, and P, i.e., if $f: M'_R \to M_R$, $g: {}_RN'_S \to {}_RN_S$, $h: P_S \to P'_S$ are hom, and

$$\left[\begin{array}{l} \alpha: \operatorname{Hom}_S(M\otimes_R N, P) \xrightarrow{\sim} \operatorname{Hom}_R(M, \operatorname{Hom}_S(N,P)), \\ \alpha': \operatorname{Hom}_S(M'\otimes_R N', P') \xrightarrow{\sim} \operatorname{Hom}_R(M', \operatorname{Hom}_S(N',P')) \end{array}\right]$$

are as in Thm 5.10, then the following diagram commutes:

$$\begin{tikzcd} \operatorname{Hom}_S(M\otimes_R N, P) \arrow[r, "\alpha", "\sim"'] \arrow[d, "\operatorname{Hom}(f\otimes g, h)"] \arrow[dr, phantom, "/\!/\!/"] & \operatorname{Hom}_R(M, \operatorname{Hom}_S(N,P)) \arrow[d, "\operatorname{Hom}(f, \operatorname{Hom}(g,h))"] \\ \operatorname{Hom}_S(M'\otimes_R N', P') \arrow[r, "\alpha'", "\sim"'] & \operatorname{Hom}_R(M', \operatorname{Hom}_S(N',P')) \end{tikzcd}$$

use α, α'

•) $-\otimes_R N$ is a left adjoint of $\operatorname{Hom}_S(N,-)$

This immediately (by some category theory) implies that $-\otimes_R N$ commutes with colimits, in particular, that $-\otimes_R N$ is right exact!

•) Similarly: For ${}_SM_R$, ${}_RN$, ${}_SP$,

$$\operatorname{Hom}_S(M\otimes_R N, P) \cong \operatorname{Hom}_R(N, \underbrace{\operatorname{Hom}_S(M,P)}_{\text{left } R\text{-module: } rf(m)=f(mr)}) \quad \text{(as groups)}$$

i.e. $M\otimes_R -$ is left adjoint to $\operatorname{Hom}_S(M,-)$.

Thm 1.11 ($\otimes$ of free modules) Let R be a ring, M_R, ${}_RN$ modules.

(1) If ${}_RN$ is free with basis $(y_j)_{j\in J}$, then every $x \in M\otimes_R N$ has a unique representation

$$x = \sum_{j\in J}{}' m_j \otimes y_j \quad \text{where } \forall j\in J: m_j \in M \text{ and } \forall' j \in J: m_j = 0$$

(2) If M_R is free with basis $(x_i)_{i\in I}$, ${}_RN$ free with basis $(y_j)_{j\in J}$, then $M\otimes_R N$ is a free right [left] R-module with basis $\{x_i \otimes y_j \mid i\in I, j\in J\}$.

(Note: Since ${}_RR_R$ is a bimodule, so is every direct sum of it!

This gives an (R,R)-bimodule structure on ${}_RM$ and N_R, e.g.

$$\underset{R}{\overset{\cap}{r}}\Big(\sum_{i\in I}{}' x_i \underset{R}{\overset{\cap}{\lambda_i}}\Big) = \sum_{i\in I}{}' \underbrace{x_i}_{\text{basis element}} (r\lambda_i) \quad)$$

CAREFUL: $rx_i = x_i r \ \forall r\in R, i\in I$ but $rm \neq mr$ in general!

Proof: 1) $\varphi: \begin{cases} {}_R R^{(J)} \xrightarrow{\sim} {}_R N \\ (r_j)_{j\in J} \longmapsto \sum_{j\in J} r_j y_j \end{cases}$ is an R-module iso

Let $e_j \in R^{(J)}$ denote the element with a 1 in the j-th coordinate, and 0 elsewhere.

By Prop 1.7 & 1.9,

$$\Psi: M^{(J)} \xrightarrow{\sim} (M\otimes_R R)^{(J)} \xrightarrow{\sim} M\otimes_R ({}_R R^{(J)}) \xrightarrow{\sim} M\otimes_R N$$
$$(m_j)_{j\in J} \longmapsto (m_j\otimes 1)_{j\in J} \longmapsto \sum_{j\in J} m_j\otimes e_j \longmapsto \sum_{j\in J} m_j\otimes y_j$$

Since every element $m\in M^{(J)}$ has a unique repr. $m=(m_j)_{j\in J}$ with $m_j\in M$, almost all zero, the fact that Ψ is an isomorphism implies that every $x\in M\otimes_R N$ has a unique representation as claimed.

2) We show the claim for left modules

$$M\otimes_R N \cong \Big(\bigoplus_{i\in I} x_i R\Big)\otimes_R \Big(\bigoplus_{j\in J} R y_j\Big) \cong \bigoplus_{\substack{i\in I\\ j\in J}} (x_i R\otimes R y_j)$$ (as ~~abelian groups~~ R-left [right] modules)

(Since M is an (R,R)-bimodule, $M\otimes_R N$ is a left R-module.
Since $x_iR\cong R$, also x_iR is a left R-bimodule
(One checks that the iso above preserves the left R-module structure)

$\forall i\in I, j\in J$:

$${}_R R \xrightarrow{\sim} {}_R R\otimes_R R \xrightarrow{\sim} {}_R(x_iR\otimes_R Ry_j)$$
$$r \longmapsto r\otimes 1 \longmapsto x_i r\otimes y_j = r(x_i\otimes y_j)$$

only for basis!

$\Rightarrow$ $M\otimes_R N$ is a free left R-module with basis $(x_i\otimes y_j)_{i\in I, j\in J}$.

$\square$

Cor 1.12: Let R be a commutative ring, and M_R, N_R free modules. Then $M\otimes_R N$ is free and $\operatorname{rank}(M\otimes_R N) = \operatorname{rank}(M)\operatorname{rank}(N)$.

Def 1.13 Let R, S be rings, $\varphi: R \to S$ a ring hom., ${}_R M$ a module.
Then ${}_S S_R$ is a (S,R)-bimodule by means of $s \cdot r := s\varphi(r)$ $\forall s \in S, r \in R$,
and ${}_S S \otimes_R M$ is an S-module.
${}_S S \otimes_R M$ is the base extension (or scalar extension) of M to S.

Prop 1.14 (1) If ${}_R M$ is free with basis X, then $S \otimes_R M$ is free with basis $\{1_S \otimes x \mid x \in X\}$.

(2) If ${}_R M$ is projective, then $S \otimes_R M$ is projective.
(3) — " — finitely generated, —— " —— finitely generated.

Proof (1) Follows from ${}_S S \otimes_R M \cong {}_S S \otimes_R (R^{(X)}) \cong ({}_S S \otimes_R R)^{(X)} \cong ({}_S S_R)^{(X)}$

(2) ${}_R M$ projective $\to \exists\, {}_R N$, free module ${}_R F$: ${}_R M \oplus {}_R N \cong {}_R F$

$$\Longrightarrow ({}_S S \otimes_R M) \oplus_S ({}_S S \otimes_R M) \cong {}_S S \otimes_R ({}_R M \oplus {}_R N) \cong \underbrace{{}_S S \otimes_R F}_{\text{free } S\text{-module, by (1)}}$$

$\Longrightarrow$ ${}_S S \otimes_R M$ is a projective S-module

(3) ${}_R M$ f.g. $\Rightarrow \exists n \in \mathbb{N}_0$, epi ${}_R R^n \twoheadrightarrow {}_R M$

$\overset{S\otimes_R-\text{ right exact}}{\Longrightarrow}$ ${}_S S \otimes_R R^n \twoheadrightarrow {}_S S \otimes_R M$ epi $\Longrightarrow$ $S \otimes_R M$ f.g. $\square$

Reminder: ${}_R M$ is projective if it satisfies the following equivalent conditions:
(a) ${}_R M$ is a direct summand of a free module
(b) Every exact sequence $0 \to {}_R K \to {}_R L \to {}_R M \to 0$ splits.
(c)

$$\begin{tikzcd} & K \arrow[dl, dashed, "g"'] \arrow[d, "f"] & \\ {}_R L \arrow[r, "p"'] & {}_R M \arrow[r] & 0 \end{tikzcd}$$

For every epi $p: L \twoheadrightarrow M$, and hom. $f: {}_R K \to {}_R M$, $\exists g: K \to L$ s.t. $p \circ g = f$.

(d) The functor $\mathrm{Hom}_R({}_R M, -): R\text{-Mod} \to \mathrm{Ab}$ is exact
(e) [dual basis] $\exists\, (a_i)_{i \in I}$ in M and $(f_i)_{i \in I}$ in M^* s.t. $(\forall x \in M\ \forall' i \in I: f_i(x) = 0)$ and
$x = \sum_{i \in I}' f_i(x) a_i$

($\mathrm{End}(P) = P^* P$)

Reminder: Let R be a commutative ring. An ring A is an R-algebra if

- $(A,+)$ is an R-module, and
- $\forall r \in R \forall a,b \in A$: $r(ab) = (ra)b = a(rb)$.

Equivalently, an algebra A can be described by a ring hom $\varphi: R \to Z(A)$.
$r \mapsto r \cdot 1_A$

Tacitly, we call $\varphi: R \to A$ or A an algebra, interchangeably.

Remarks & Examples (1) Let R be a commutative ring, $\varphi: R \to A$, $\psi: R \to B$ algebras.
Then $A \otimes_R B$ is an R-algebra with $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ for $a,a' \in A$, $b,b' \in B$ and structure hom. $R \to A \otimes_R B$, $r \mapsto \varphi(r) \otimes 1 = 1 \otimes \psi(r)$

$A \otimes_R B$ is the coproduct of A and B in the category of R-algebras.

(2) If R, S are commutative rings, $\varphi: R \to S$, $\psi: R \to B$ are algebras, then
$S \to S \otimes_R B$, $s \mapsto s \otimes 1$ is an S-algebra.
[Note: $(s \otimes 1)(s' \otimes b') = ss' \otimes b' = (s' \otimes b')(s \otimes 1)$.]

(3) Let R be a commutative ring, $\varphi: R \to A$ an algebra.
In particular: $M_m(R) \otimes M_n(R) \cong M_{mn}(R)$

$A \otimes_R M_n(R) \cong M_n(A)$ $\forall$, $A \otimes_R R[X] \cong A[X]$ (as rings and as A-modules)

$\uparrow$ free R-module with basis $\{e_{i,j} \mid i,j \in [1,n]\}$, $\sum_{i=1}^{n} e_{i,i} = 1_{M_n(R)}$, $e_{i,j} e_{k,\ell} = \delta_{j,k} e_{i,\ell}$, ($R$ is centralizer of $\{e_{i,j}\}$)

$\uparrow$ UP of $S[X]$

(4) Let R be a ring, and $S \subset R$ a multiplicatively closed set of regular elts satisfying the right Ore condition ($\forall r \in R \forall s \in S$: $rS \cap sR \neq \emptyset$)
Then a right quotient ring RS^{-1} exists, and for every M_R we can define MS^{-1}. We have: $MS^{-1} \cong M \otimes_R RS^{-1}$.

Def. 1.15 (Tensor algebra) Let R be a commutative ring, ${}_RV$ a module,
$V^{\otimes 0} := R$, $V^{\otimes n} := V^{\otimes(n-1)} \otimes V$ for $n \geq 1$.

$\forall m,n \geq 0$: $\exists$ R-hom:
$$\mu_{m,n}: \begin{cases} V^{\otimes m} \otimes_R V^{\otimes n} \longrightarrow V^{\otimes(m+n)} \\ (v_1 \otimes \_ \otimes v_m) \otimes (v_{m+1} \otimes \_ \otimes v_n) \longmapsto v_1 \otimes \_ \_ \otimes v_m \otimes v_{m+1} \otimes \_ \otimes v_n \end{cases}$$

Define $T(V) := \bigoplus_{m \geq 0} V^{\otimes m}$.

The family of maps $(\mu_{m,n})_{m,n\geq 0}$ induces an R-bilinear map
$$\begin{cases} T(V) \times T(V) \longrightarrow T(V) \\ \Big(\sum_{m\geq 0} \underset{\substack{\cap \\ V^{\otimes m}}}{x_m}, \sum_{n \geq 0} \underset{\substack{\cap \\ V^{\otimes n}}}{y_n}\Big) \longmapsto \sum_{m,n \geq 0} \mu_{m,n}(x_m \otimes y_n) \end{cases}$$
With this operation $T(V)$ is an R-algebra, called the tensor algebra of V.

Remarks (1) $T(V)$ is a ($\mathbb{Z}$-) graded algebre with $T^m(V) := V^{\otimes m}$ for $m \geq 0$, $T^m(V) = 0$ for $m < 0$. [I.e. $V^{\otimes m} V^{\otimes n} \subseteq V^{\otimes(m+n)}$.]

$I \trianglelefteq T(V)$ is homogeneous if it satisfies the following equivalent conditions;
(a) I is generated by homogeneous elts (elts of $\bigcup_{m\geq 0} V^{\otimes m}$)
(b) $\forall x \in I$ the homogeneous components of x belong to I.

Then $T(V)/_I \cong \bigoplus_{m \geq 0} (T^m(V) + I)/_I$ is also graded, with m-th homogeneous compon. $T^m(V) + I/_I$.

(2) $\mathrm{Sym}(V) := T(V)/\langle\{v \otimes w - w \otimes v \mid v, w \in V\}\rangle \cong \bigoplus_{m \geq 0} \mathrm{Sym}^m(V)$
is the symmetric algebra on V

(3) $\Lambda(V) := T(V)/\langle\{v \otimes v \mid v \in V\}\rangle \cong \bigoplus_{m\geq 0} \Lambda^m(V)$ is the exterior algebra (or Grassmann algebra) on V.

(4) Suppose V is free with basis X.

Then: $T(V) \cong R\langle \underline{X} \rangle$, $\mathrm{Sym}(V) \cong R[\underline{X}]$, and
$$\Lambda(V) = R\langle \underline{X} \rangle / \langle\{x^2 = 0,\ xy + yx = 0 \mid x, y \in X\}\rangle$$

[We will meet $\Lambda(V)$ again later in the course, and discuss it in some more detail]

(5) Let R be a ring, ${}_RM$ a module, $I \trianglelefteq_r R$ a right ideal.

$IM := \langle \{xm \mid x \in I, m \in M\} \rangle_{\mathbb{Z}}$ is a subgroup of M.

If $I \trianglelefteq R$, then IM is a R-submodule of M.

Claim: $M/IM \cong R/I \otimes_R M$ (as abelian groups, as R-modules if $I \trianglelefteq R$)

Proof: $\exists$ group epi $\mu_I : \begin{cases} I \otimes_R M \longrightarrow IM \\ x \otimes m \longmapsto xm \end{cases}$ (multiplication homomorphism)

From the SES $0 \to I \xhookrightarrow{i} R \xrightarrow{\pi} R/I \to 0$, we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
 & I \otimes_R M \arrow[r, "i \otimes M"] \arrow[d, "\mu_I"] \arrow[dr, phantom, "\cong"] & R \otimes_R M \arrow[r, "\pi \otimes M"] \arrow[d, "\varphi", "\wr"'] & R/I \otimes_R M \arrow[r] & 0 \\
0 \arrow[r] & IM \arrow[r] & M \arrow[r] \arrow[ur, bend left, "p"] & M/IM \arrow[r] & 0
\end{tikzcd}$$

$\varphi(r \otimes m) = rm$

$p = (\pi \otimes M) \circ \varphi^{-1} : M \longrightarrow R/I \otimes_R M$ is an epi, $\ker(p) = \mu_I(I \otimes_R M) = IM$

$\Longrightarrow p$ induces iso $M/IM \xrightarrow{\sim} R/I \otimes_R M$, $m + IM \longmapsto (1+I) \otimes m$.

¥ (17 ½)

2. Flat modules

Def 2.1 Let R be a ring. A module ${}_RN$ is flat if the functor $- \otimes_R N : \text{Mod-}R \longrightarrow Ab$ is exact.

Remark: Since $- \otimes_R N$ is right exact,

${}_RN$ is flat $\Longleftrightarrow$ For every monomorphism $i : M'_R \to M_R$, the hom. $i \otimes N : M' \otimes_R N \to M \otimes_R N$ is a mono.

Prop 2.2 Let R be a ring.

(1) Let $({}_RN_i)_{i \in I}$ be a family of modules and ${}_RN = \bigoplus_{i \in I} N_i$.

N is flat $\Longleftrightarrow$ $\forall i \in I$: N_i is flat.

(2) Every projective module is flat.

Proof: (1) Let $f: K_R \to L_R$ be a monomorphism.
Then there is a commutative diagram

$$\begin{tikzcd} K \otimes_R N \arrow[r, "f\otimes N"] \arrow[d, "\wr"'] \arrow[dr, phantom, "\#"] & L \otimes_R N \arrow[d, "\wr"] \\ \bigoplus_{i\in I} (K \otimes_R N_i) \arrow[r, "(f\otimes N_i)_{i\in I}"] & \bigoplus_{i\in I} (L \otimes_R N_i) \end{tikzcd}$$

Hence, $f\otimes N$ is injective if and only if $\forall i\in I$: $f\otimes N_i$ is injective.
From this, the claim is immediate.

(2) Let $f: K_R \to L_R$ be a monomorphism.
Then

$$\begin{tikzcd} & K \otimes_R R \arrow[r, "f\otimes R"] \arrow[d, "\wr"] \arrow[dr, phantom, "\#"] & L \otimes_R R \arrow[d, "\wr"] \\ 0 \arrow[r] & K \arrow[r, "f"] & L \end{tikzcd}$$

(exact) implies that $_R R$ is flat.

$\overset{(1)}{\Longrightarrow}$ Every free module is flat $\overset{(1)}{\Longrightarrow}$ Every projective module is flat. □

Example: (2) $_{\mathbb{Z}}\mathbb{Q}$ is flat but not projective.

- $_{\mathbb{Z}}\mathbb{Q}$ is flat since $_{\mathbb{Z}}\mathbb{Q} = _{\mathbb{Z}}(\mathbb{Z}\setminus\{0\})^{-1}\mathbb{Z}$.

- $_{\mathbb{Z}}\mathbb{Q}$ is not projective: projective modules over a PID are free (as submodule of free module), and $\mathbb{Q}_{\mathbb{Z}}$ is obviously not free, since any two non-zero elements are linearly dependent)

(1) Let R be a ring, and $S\subset R$ a multiplicatively closed set of regular elements set satisfying the right Ore condition.
Then $_R(RS^{-1})$ is flat since $M\otimes_R (RS^{-1}) \cong MS^{-1}$ and localization of modules is exact.

Lemma 2.3: Let R be a ring, ${}_R N$ module, $\mu: L_R \longrightarrow M_R$ monomorphism. If $\forall$ f.g. $L'_R \subseteq L_R$, $\mu|_{L'} \otimes L' : L' \otimes_R N \longrightarrow M \otimes_R N$ is injective, then $\mu \otimes L : L \otimes_R N \longrightarrow M \otimes_R N$ is injective.

Proof. Suppose $\exists x \in L' \otimes_R N \setminus \{0\}$: $\mu \otimes L(x) = 0$.

$$\longrightarrow x = \sum_{i=1}^{k} l_i \otimes n_i, \quad l_i \in L, \ n_i \in N$$

$L' := \langle l_1, \dots, l_k \rangle_R$, let $j: L' \hookrightarrow L$,

$$x' := \sum_{i=1}^{k} l_i' \otimes n_i$$

$0 \neq x = j \otimes N(x') \Rightarrow x' \neq 0 \quad \mu|_{L'} \otimes N = (\mu \circ j) \otimes N(x') = (\mu \otimes N) \circ (j \otimes N)(x') = 0 \quad \lightning.$ $\square$

Thm 2.4 (Flat test) Let R be a ring, ${}_R N$ a module. Then N is flat if and only if, for every f.g. right ideal $I \trianglelefteq_r R$ the multiplication hom. $\mu_I: I \otimes_R R \longrightarrow I$, $x \otimes r \longmapsto xr$ is injective.

Proof: „$\Rightarrow$" ✓, „$\Leftarrow$" Show: $\forall$ mono: $j: K_R \hookrightarrow M_R$, $j \otimes N$ injective.

W.l.o.g.: $j: K_R \hookrightarrow M_R$ embedding of submod $K_R \subseteq M_R$,

since:
$$\begin{array}{ccc} K_R & \xhookrightarrow{j} & M_R \\ {\scriptstyle \wr}\downarrow j & & \| \\ j(K_R) & \underset{\subseteq}{\hookrightarrow} & M_R \end{array}$$

Step 1: $M_R = R^n_R$, $n > 0$. By induction.

$n=0$ ✓, $n=1$: $K \trianglelefteq_r R \xRightarrow[\text{Lemma 2.3}]{\text{assumpt. \&}} j \otimes N$ injective.

$n>1$, $n-1 \to n$: $R^{n-1} \subseteq R^n$ (embedded in 1^{st} $n-1$ components)

$$K/_{K \cap R^{n-1}} \cong K+R^{n-1}/_{R^{n-1}} \subseteq R^n/_{R^{n-1}} \cong R_R \Rightarrow \exists \text{mono } j: K+R^{n-1}/_{R^{n-1}} \hookrightarrow R_R$$

$\exists$ commutative diagram with exact rows

$$\begin{tikzcd}
 & (K\cap R^{n-1})\otimes N \arrow[r, "g"] \arrow[d, "i"] & K\otimes N \arrow[r, "h"] \arrow[d, "f"] & \frac{K+R^{n-1}}{K}\otimes N \arrow[r] \arrow[d, "j\otimes N"] & 0 \\
0 \arrow[r] & R^{n-1}\otimes N \arrow[r, "k"] & R^n\otimes N \arrow[r, "l"] & R\otimes N \arrow[r] & 0
\end{tikzcd}$$

(split exact)

By IH, $j\otimes N$ and i are injective.

$\Longrightarrow$ f injective.

[Let $x\in K\otimes N$, $f(x)=0 \Longrightarrow 0=l\circ f(x)=(j\otimes N)\circ h(x) \Longrightarrow h(x)=0$

$\Rightarrow \exists y\in (K\cap R^{n-1})\otimes N: \quad g(y)=x.$

$\Rightarrow k\circ i(y)=f\circ g(y)=0 \overset{k,i\text{ inj.}}{\Longrightarrow} y=0 \Rightarrow x=0.$]

Step 2: M_R free with basis $(e_i)_{i\in I}$.

Wlog. (Lemma 23) K_R f.g., $K_R=\langle x_1,\dots,x_n\rangle_R$,

$\forall i\in[1,n]: \exists I_i\subseteq I, |I_i|<\infty: \quad x_i=\sum_{j\in I_i}' e_j\lambda_j,\ \lambda_j\in R.$

$J:=\bigcup_{i\in[1,n]} I_i$, $\quad F_R:=\langle e_j \mid j\in J\rangle$, is f.g. free, $K_R\subseteq F_R$, $M_R=F_R\oplus L_R$,

with a free model L_R.

$j: K_R \overset{f}{\hookrightarrow} F_R \overset{g}{\hookrightarrow} M_R$, $\quad j\otimes N=(g\otimes N)\circ(f\otimes N)$

$(f\otimes N)$ injective by step 1, $\quad (g\otimes N)$ injective because

$0\to F\xrightarrow{g} M\to L\to 0$ is split exact

Step 3: M_R arbitrary.

Let F_R be a free module s.t. $\exists p: F_R \twoheadrightarrow M_R$ epi, $L=\ker(p)$

$\exists$ comm. diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, hook, "\bar{i}"] \arrow[d, equal] & p^{-1}(K) \arrow[r, "\bar{p}"] \arrow[d, hook, "\bar{j}"] & K \arrow[r] \arrow[d, hook, "j"] & 0 \\
0 \arrow[r] & L \arrow[r, hook, "i"] & F \arrow[r, "p"] & M \arrow[r] & 0
\end{tikzcd}$$

apply $-\otimes N$:

$$\begin{tikzcd} L\otimes N \arrow[r, "\bar{i}\otimes N"] \arrow[d, equal, "id"] & \bar{p}^{-1}(K)\otimes N \arrow[r, "\bar{p}\otimes N"] \arrow[d, "\bar{j}\otimes N"] & K\otimes N \arrow[r] \arrow[d, "j\otimes N"] & 0 \\ L\otimes N \arrow[r, "i\otimes N"] & F\otimes N \arrow[r, "p\otimes N"'] & M\otimes N \arrow[r] & 0 \end{tikzcd}$$

$\bar{j}\otimes N$ is injective (Step 2), id inj.

$\Longrightarrow$ $j\otimes N$ injective

[Let $x\in K\otimes N$: $j\otimes N(x)=0$

$\longrightarrow \exists y\in \bar{p}^{-1}(K)\otimes N$: $x=(\bar{p}\otimes N)(y)$ $\Longrightarrow$ $(p\otimes N)\circ\overbrace{(\bar{j}\otimes N)(y)}^{\in \mathrm{Ker}(p\otimes N)}=0$

$\Longrightarrow$ $\exists z\in L\otimes N$: $\bar{j}\otimes N(y)=i\otimes N(z)=(\bar{j}\otimes N)\circ(\bar{i}\otimes N)(z)$

$\overset{\bar{j}\otimes N \text{ inj}}{\Longrightarrow}$ $y=\bar{i}\otimes N(z)$ $\Longrightarrow x=\bar{p}\otimes N(y)=0$]

$\square$

II. Polynomial Identity Rings

0. Free algebras

For a set X, let $\langle X\rangle$ denote the free (non-abelian) monoid on X. The elements of $\langle X\rangle$ are words $x_{i_1}\cdots x_{i_k}$, $x_{i_j}\in X$ and $1\in\langle X\rangle$ denotes the empty word. The multiplication is defined by concatenation of words.

Def. 0.1 Let A be a commutative ring and X a set.
The free (A-) algebra on the set X, $A\langle X\rangle$, is the free A-module with basis $\langle X\rangle$ and multiplication defined by A-linear extension of the multiplication on $\langle X\rangle$.
Elements of $A\langle X\rangle$ are called polynomials in the noncommuting indeterminates $\langle X\rangle$. Elements of $\langle X\rangle$ are monomials. Every $f\in A\langle X\rangle$ has a unique representation

$$f=\sum_{w\in\langle X\rangle} a_w w\ ,\quad a_w\in A,\quad \forall' w\in\langle X\rangle:\ a_w=0,$$

and $\operatorname{supp}(f)=\{w\in\langle X\rangle \mid a_w\neq 0\}$ is the support of f.

Lemma 0.2 (UP of the free algebra) Let A be a commutative ring, X a set, R an A-algebra, and for all $x\in X$ let $r_x\in R$.
Then there exists a unique A-algebra hom $\varphi: A\langle X\rangle\to R$ s.t. $\varphi(x)=r_x$ for all $x\in X$.

Remark: • $A\langle X\rangle$ is a semigroup algebra.
• If F_A is the free A-module with basis X, then $A\langle X\rangle\cong T(F)$, where $T(F)$ is the tensor algebra.

NOTATION: In Ch. II, A denotes a commutative ring, $A\langle x_1,\dots,x_n\rangle$ the free algebra in n indeterminates $x_1,\dots,x_n$, and $A\langle X\rangle$ the free algebra in countably many indeterminates $x_1, x_2, \dots$.
(sometimes x, y, z, y_i, z_i, as is convenient)
Any $f \in A\langle X\rangle$ only involves finitely many indeterminates, and we write $f = f(x_1,\dots,x_n)$ to emphasize $f \in A\langle x_1,\dots,x_n\rangle \subseteq A\langle X\rangle$.
(*)

Def 0.3 (1) $A\langle X\rangle$ is ($\mathbb{Z}$-)graded with $\deg(x) = 1$ for all $x \in X$.
$f \in A\langle X\rangle$ is homogeneous of degree d if it is an A-linear combination of monomials of degree d.

(2) $A\langle X\rangle$ is ($\mathbb{Z}^{|X|}$-) multigraded, where x_i has degree 1 in the i-th component. ($\deg_{x_i}(x_i) = 1$, $\deg_{x_i}(x_j) = 0$ for $i \neq j$)
$f \in A\langle X\rangle$ is linear in x_i if each monomial of f has degree 1 in x_i.
— " — multilinear in $x_{i_1},\dots,x_{i_m}$ — " — in x_{i_j} for all $j \in [1,m]$.
$f \in A\langle X\rangle$ is multilinear if it is linear in each of its variables.

Note: $f(x_1,\dots,x_n)$ is multilinear in $x_1,\dots,x_n$ $\iff$

$$f = \sum_{\sigma \in \mathfrak{S}_n} a_\sigma \, x_{\sigma(1)} \cdots x_{\sigma(n)} \quad \text{with } a_\sigma \in A \text{ for } \sigma \in \mathfrak{S}_n.$$

$\mathfrak{S}_n$ acts on multilinear polynomials in $x_1,\dots,x_n$ by

$$(\tau f)(x_1,\dots,x_n) = f(x_{\tau(1)},\dots,x_{\tau(n)}) = \sum_{\sigma \in \mathfrak{S}_n} a_\sigma \, x_{\tau\sigma(1)} \cdots x_{\tau\sigma(n)} \quad \text{for } \tau \in \mathfrak{S}_n.$$

Def 0.4 For $n \in \mathbb{N}_0$,

$$S_n = \sum_{\sigma \in \mathfrak{S}_n} (\operatorname{sgn}\sigma)\, x_{\sigma(1)} \cdots x_{\sigma(n)}$$

is the n-th standard identity.

(*) If $f(x_1,\dots,x_n) \in A\langle X\rangle$ and $r_1,\dots,r_n \in R$, then $f(r_1,\dots,r_n)$ denotes the substitution of r_i for x_i.
[Let $\varphi: A\langle X\rangle \to R$ be any hom. with $\varphi(x_i) = r_i$ for $i \in [1,n]$. Then $f(r_1,\dots,r_n) := \varphi(f)$]

1. Polynomial Identities

Def 1.1 Let R be an A-algebra.

(1) $f \in A\langle X\rangle$ (sometimes: „$f=0$") is a polynomial identity (PI) for R if $f(r_1, \dots, r_n) = 0$ for all $r_1, \dots, r_n \in R$.
We also say: „R satisfies f [$f=0$]"

(2) $f \in A\langle X\rangle$ is monic if some coefficient in the highest degree homogeneous component is 1.

(3) R is a polynomial identity algebra over A (PI-algebra) if there is a monic $f \in A\langle X\rangle$ s.t. f is a PI for R.
If $A = \mathbb{Z}$, we say R is a PI ring.

Remarks: ·) If K is a field, char $K = p > 0$, then only 0 is a PI for $K\langle X\rangle$ over K, but $0 = px_1$ is a PI for $K\langle X\rangle$ over $\mathbb{Z}$.
Since $K\langle X\rangle$ should not be a PI ring, we require a PI ring to have a monic PI.
[Highest coeff. to make multilinearization work.)

·) Let R be an A-algebra. If R is a PI ring, then it is also a PI algebra over A. The converse is true but nontrivial (Amitsur, 1952)

Examples: Let R be an A-algebra.

(1) If R is commutative, it satisfies $[x,y] = xy - yx$

(2) A boolean ring satisfies $x^2 - x$ (by Def.)

(3) $R = \mathbb{F}_q$ with $q = p^n$, $p \in \mathbb{P}$, $n \in \mathbb{N}$, satisfies $x^q - x$.

(4) $R = M_2(A)$ satisfies the Hall identity $[[x,y]^2, z]$

Proof: ·) $B \in M_2(A)$ satisfies $B^2 - \mathrm{tr}(B)B + \det(B) = 0$ (Cayley-Hamilton)
·) If $\mathrm{tr}(B) = 0$ then $B^2 = -\det(B)$ is central in $M_2(A)$
·) $\mathrm{tr}(BC) = \mathrm{tr}(CB)$ for $B, C \in M_2(A) \Rightarrow \mathrm{tr}([B,C]) = \mathrm{tr}(BC - CB) = 0$

$\longrightarrow \forall B,C \in M_2(A)$: $[B,C]^2$ is central $\Rightarrow \forall B,C,D \in M_2(A)$: $[[B,C]^2, D]=0$

(5) Let $R=T_n(A)$ be the ring of $n\times n$ upper triangular matrices

$\forall B,C \in T_n(A)$: $[B,C] \in T_n(A)$ has 0 in all diagonal entries

$\longrightarrow [x_1,y_1][x_2,y_2]\cdots[x_n,y_n]$ is a PI for $T_n(A)$.

(6) $A\langle X\rangle$ does not satisfy a PI: If $f=f(x_1,\dots,x_n) \in A\langle X\rangle$ vanishes on $A\langle X\rangle$, then in particular $0=f=f(x_1,\dots,x_n)$.

Lemma 1.2 (1) Subalgebras and homomorphic images of PI-algebras are PI-algebras.

(2) If $R_1,\dots,R_n$ are PI-algebras, then so is $R=\prod_{i=1}^n R_i$

(3) If $(R_i)_{i\in I}$ is a family of PI-algebras satisfying the same PI $f\in A\langle X\rangle$, then $R=\prod_{i\in I} R_i$ is a PI-algebra satisfying f.

(4) Let R be an A-algebra. If $N \trianglelefteq R$ is nilpotent and R/N is a PI algebra, then so is R.

Proof: (1) ✓

(2) If $f=f(x_1,\dots,x_m) \in A\langle X\rangle$ and $\forall j\in[1,m]$: $\underline{r}^{(j)}=(r_1^{(j)},\dots,r_n^{(j)}) \in R$,

then $f(\underline{r}^{(1)},\dots,\underline{r}^{(m)}) = \left(f(r_1^{(1)},\dots,r_1^{(m)}),\dots,f(r_n^{(1)},\dots,r_n^{(m)})\right)$

Let f_i be a monic PI for R_i. Then $f=f_1\cdots f_n$ is a monic PI for R

(3) ✓ Also (2)

(4) If f is a monic PI for R/N and $k\in\mathbb{N}_0$: $N^k=0$

Then f^k is a monic PI for R.

Def 1.3 $I \trianglelefteq A\langle X\rangle$ is a *T-ideal* if $\varphi(I) \subseteq I$ for every $\varphi \in \mathrm{End}(A\langle X\rangle)$.
Given $I \trianglelefteq A\langle X\rangle$, we denote by I^T the smallest T-ideal containing I.
Note that endomorphisms of $A\langle X\rangle$ are given by substitutions $x_i \mapsto f_i$, $f_i \in A\langle X\rangle$.

Def. & Thm 1.4 Let R be an A-algebra, and

$$T(R) := \{ f(x_1,\dots,x_n) \in A\langle X\rangle \mid \forall r_1,\dots,r_n \in R: f(r_1,\dots,r_n) = 0\}$$
$$\overset{(*)}{=} \bigcap \{\mathrm{Ker}(\varphi) \mid \varphi: A\langle X\rangle \to R \text{ } A\text{-alg. hom.}\}$$

is a T-ideal of $A\langle X\rangle$, called the T-ideal of identities of R.
If $J \trianglelefteq A\langle X\rangle$ is a T-ideal, then $T(A\langle X\rangle/J) = J$. If J contains a monic polynomial, $A\langle X\rangle/J$ is a PI-algebra.

Proof. $(*)$. „$\subseteq$": Let $\varphi: A\langle X\rangle \to R$, $f = f(x_1,\dots,x_n) \in T(R)$
$\Rightarrow \varphi(f) = \varphi(f(x_1,\dots,x_n)) = f(\varphi(x_1),\dots,\varphi(x_n)) = 0$
„$\supseteq$": Let $f = f(x_1,\dots,x_n) \in \bigcap\{\ker\varphi \mid \varphi: A\langle X\rangle \to R\}$, and $r_1,\dots,r_n \in R$.
By the UP (Lemma 0.2), $\exists \varphi: A\langle X\rangle \to R$: $\varphi(x_i) = r_i$ for $i \in [1,n]$
$\Rightarrow 0 = \varphi(f) = f(\varphi(x_1),\dots,\varphi(x_n)) = f(r_1,\dots,r_n) \rightarrow f \in T(R)$

$T(R)$ is a T-ideal ✓

$T(A\langle X\rangle/J) = J$: „$\subseteq$" Let $\pi: A\langle X\rangle \to A\langle X\rangle/J$, $g \mapsto g + J =: \bar{g}$
Let $f(x_1,\dots,x_n) \in T(A\langle X\rangle/J)$ $\overset{*}{\Longrightarrow} f \in \mathrm{Ker}\,\pi = J$
„$\supseteq$": Let $f = f(x_1,\dots,x_n) \in J$, $r_1,\dots,r_n \in A\langle X\rangle/J$
$\Rightarrow \exists m \geq n\ \exists g_1(x_1,\dots,x_m),\dots,g_n(x_1,\dots,x_m) \in A\langle X\rangle\ \forall i \in [1,n]: r_i = \overline{g_i(x_1,\dots,x_m)}$
J T-ideal $\Rightarrow f(g_1(x_1,\dots,x_m),\dots,g_n(x_1,\dots,x_m)) \in J$
$\Rightarrow f(r_1,\dots,r_n) = f(\bar{g}_1,\dots,\bar{g}_n) = \overline{\underbrace{f(g_1,\dots,g_n)}_{\in J}} = 0$ $\square$

Some of the main questions in PI theory concern the study of $T(R)$:

- Is $T(R)$ always f.g. as a T-ideal?
  (I.e., are all identities consequences of finitely many identities?)
  (Specht Problem)

  Yes, when A is a field of characteristic 0 (Kemer, '87)
  No, when — " — >0
  (Belov '99, Grishin '99, Shchigolev '99, now many examples)

- For a given R, find generators for $T(R)$:
  - E.g. if $A=K$ is a field, V_K ∞-dim vector space,
    $T(\Lambda(V)) = \langle [x,y,z] \rangle^T$
    $[[x,y],z]$
  - Only few other examples known, e.g. $T_n(K)$, $M_2(K)$
    $[x_1,y_1]\cdots[x_n,y_n]$; s_4 and $[[x,y]^2,x]$

2. Multilinear Identities

<u>Thm 2.1</u> If $f \in A\langle X\rangle$ is a PI of R with $\deg(f)=d$, then R also satisfies a multilinear identity g such that: $\deg(g) \le \deg(f)$, each coefficient in g is also a coefficient of f, and if f is monic, so is g.

<u>Proof:</u> <u>Claim A</u>: Can assume all monomials in f have the same support.

<u>Proof of A</u>: Let $f(x_1,\dots,x_n) = f_1(x_1,\dots,x_n) + f_2(x_2,\dots,x_n)$, where f_1 consists of all monomials of f having x_1 in the support, and f_2 consists of all the other monomials

Setting $x_1 = 0 \Rightarrow f_2$ is a PI for $R \Rightarrow f_1$ is a PI for R.

If f is monic, then also one of f_1 or f_2 will be.

Repeating this procedure for $x_2, \dots, x_n$ proves the claim. □(Claim)

Suppose now that f is as in Claim A, and not already multilinear.

$\overset{\text{Claim A}}{\Longrightarrow}$ The monomials of highest degree in $\mathrm{supp}(f)$ are not multilinear.

$\Rightarrow$ There exists an indeterminate, say x_1, which occurs with degree ≥ 2 in the monomials of highest total degree in $\mathrm{supp}(f)$

Set $g(x_1, \dots, x_{n+1}) = f(x_1 + x_{n+1}, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n) - f(x_{n+1}, x_2, \dots, x_n)$

The monomials in $\mathrm{supp}(g)$ are precisely those coming from monomials m in f with $\deg_{x_1}(m) \geq 2$. Therein, each of these monomials some — but not all — x_1's have been replaced by x_{n+1}

(E.g. $x_1 x_2 x_1 x_3 \in \mathrm{supp}(f) \rightsquigarrow x_1 x_2 x_{n+1} x_3 + x_{n+1} x_2 x_1 x_3$

$x_1^3 \rightsquigarrow x_{n+1} x_1^2 + x_1 x_{n+1} x_1 + x_{n+1} x_1^2 + x_{n+1}^2 x_1 + x_1 x_{n+1}^2 + x_{n+1} x_1 x_{n+1}$)

Clearly, the coefficients in g are coefficients in f, and if f is monic, so is g.

Since $\deg_{x_1}(g) < \deg_{x_1}(f)$, $\deg_{x_i}(g) \leq \deg_{x_i}(f)$, for $i \in [2, n]$, $\deg_{x_{n+1}}(g) < \deg_{x_1}(f)$, an induction completes the proof.

□

~~If $A = K$ is a field, char $K = 0$ we can say more:~~

~~**Thm 2.2** Let $A = K$ be a field.~~

~~Let $f = \sum_{i=0}^{d} f_i \in K\langle X \rangle$, where f_i is the homogeneous component of f of degree i in x_1.~~

~~(i) If $|K| > d$, then the PIs $f_i = 0$ follow from $f = 0$ (i.e. $f_0, \dots, f_d \in \langle f \rangle^T$)~~

~~(ii) If char $K = 0$ or char $K > \deg(f)$, $f = 0$ is equivalent to a set of multilinear PIs.~~

One can also (easily) show:

Thm 2.2 If A is a comm. $\mathbb{Q}$-alg. $\Rightarrow$ Every T-ideal in $A\langle X\rangle$ is generated by multilinear polynomials.

Lemma 2.3 Let $f(x_1,\dots,x_n) \in A\langle X\rangle$ be multilinear in $x_1,\dots,x_n$, and let M be a set of generators of R as an A-module. If $f(m_1,\dots,m_n)=0$ for all $m_1,\dots,m_n \in M$, then f is a PI for R.

Proof: Let $r_1,\dots,r_n \in R$. Then

$\forall i \in [1,n]$: $\quad r_i = \sum_{m_i \in M} a_{i,m_i} m_i$, with $a_{i,m} \in A$, almost all zero.

$$\Rightarrow f(r_1,\dots,r_n) = \sum_{m_1\in M} \cdots \sum_{m_n \in M} a_{1,m_1} \cdots a_{n,m_n} \underbrace{f(m_1,\dots,m_n)}_{=0} = 0$$

$\square$

A central extension is a ring extension $R \subseteq S$ such that, as a ring, S is generated by R and $Z(S)$.

Cor 2.4: Let S be a central extension of an A-algebra R.
(1) If R satisfies a multilinear identity $f \in A\langle X\rangle$, then so does S.
(2) If R is a PI-algebra, then so is S.

Proof: (1) Let $S_0 := Z(S)$. The structure hom $\varphi: A \to S_0 \subseteq S$ induces a hom $\overline{\varphi}: A\langle X\rangle \to S_0\langle X\rangle$.
S is an S_0-algebra generated by R, and hence Lemma 2.3 implies that $\overline{\varphi}(f)$ is an identity for S.
$\to f \in A\langle X\rangle$ is an identity for S.

(2) By Thm 2.1, R satisfies a monic multilinear identity. By (1), so does S. $\square$

Exm: If R is a PI algebra and $R[T]$ is a polynomial ring in any number of central indeterminates T, then $R[T]$ is also a PI-algebra
($R[T]$ is generated by R and $Z[T]$)

30

Def 2.5 A multilinear $f(x_1,\dots,x_n)\in A\langle X\rangle$ is alternating if

$\forall i,j\in[1,n]:\ i\neq j:\quad f(x_1,\dots,x_{j-1},x_i,x_{j+1},\dots,x_n)=0.$

[$\Leftrightarrow$ The interchange of any two indeterminates in f gives $-f$].

The standard identity $S_n=\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\,x_{\sigma(1)}\cdots x_{\sigma(n)}$ is alternating!

Lemma 2.6: Let R be an A-algebra. Let $f(x_1,\dots,x_m)\in A\langle X\rangle$ be alternating and multilinear of degree $m\geq n+1$. If $M=At_1+\dots+At_n\subseteq R$ is a A-submodule generated by n elements, and $r_1,\dots,r_m\in M$, then $f(r_1,\dots,r_m)=0$.

Proof: Every r_i is an A-linear combination of the form $\sum_{j=1}^{n}a_{ij}t_j$. Since f is multilinear, $f(r_1,\dots,r_m)$ is an A-linear combination of evaluations $f(t_{i_1},\dots,t_{i_m})$. Since $m\geq n+1$, and f is alternating, the pigeon hole principle implies $f(t_{i_1},\dots,t_{i_m})=0$. □

Def 2.7 An A-algebra R has bounded degree n over A if for each $r\in R$, there is an A-submodule $At_1+\dots+At_n\subseteq R$ with n generators such that $A[r]\subseteq M$.

Thm 2.8: Let R be an A-algebra. Let $f\in A\langle X\rangle$ be an alternating, multilinear polynomial of degree $m\geq n+1$. (E.g. $f=S_m$)

(1) If R is f.g. as an A-module by n elements, then R satisfies f. In particular R satisfies the monic PI S_m.

(2) If R has bounded degree n, then R satisfies

$S_m(x^{m-1}y,\ x^{m-2}y,\ \dots,\ xy,\ y)\ (\neq 0)$

(This is a PI in 2 indeterminates)

Proof. (1) ✓ by Lemma 2.6.

(2) Let $r, s \in R$. By assumption, there exist $t_1, \dots, t_n \in R$ such that $r^{m-1}, \dots, r^1, r^0 = 1 \in At_1 + \dots + At_n$.

Then $r^{m-1}s, r^{m-2}s, \dots, rs, s \in At_1 s + \dots + At_n s$.

Lemma 2.6 $\Rightarrow$ $s_m(r^{m-1}s, r^{m-2}s, \dots, rs, r) = 0$

Cor 2.9 ~~(3) If R is finitely generated over a commutative subring then R is a PI ring.~~ (*)

~~(1) $M_n(A)$ satisfies s_m for all $m \geq n^2+1$.~~

~~(2) If ${}_AM$ is a f.g. module then $\mathrm{End}({}_AM)$ satisfies some s_m.~~

~~(3) If A is a field, R an A-algebra, and every element of R is algebraic of degree $\leq n \in \mathbb{N}$ over A, then R is a PI-algebra.~~

Remark: One can see directly that s_m for $m > n$ is a consequence of s_n.

$$s_{n+1}(x_1, \dots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i-1} x_i \, s_n(x_1, \dots, \hat{x}_i, \dots, x_{n+1}).$$

Proof: (1) $Z(M_n(A)) = A$ and $M_n(A)$ is generated by $\{e_{ij} \mid i,j \in [1,n]\}$.

Thus Thm 2.8 implies the claim.

~~(2) Since R_A is f.g., there exists an A-module epimorphism $\Pi: A_A^n \twoheadrightarrow R_A$. ($n \geq 1$)~~

~~Let $S := \{\varphi \in \underbrace{\mathrm{End}(A_A^n)}_{\cong M_n(A)} \mid \varphi(\ker \Pi) \subseteq \ker \Pi\} \subseteq \mathrm{End}(A_A^n)$~~

~~Every $\varphi \in S$ induces a $\varphi' \in \mathrm{End}(R_A)$, s.t. $\forall x \in A_A^n$: $\varphi'(\Pi(x)) = \Pi(\varphi(x))$.~~

~~[$\Pi \circ \varphi : A_A^n \to R_A$ has $\ker \Pi \subseteq \Pi \circ \varphi$ and hence induces $\varphi' : R_A \to R_A$.]~~

~~The map $S \to \mathrm{End}(R_A)$, $\varphi \mapsto \varphi'$ is a surjective ring homomorphism.~~

~~Finally, note that $R \cong \mathrm{End}(R_R) \subseteq \mathrm{End}(R_A)$ is a subring.~~

~~Thus, R is isomorphic to a subquotient of $M_n(A)$, and hence a PI ring.~~ □

Cor: If a ring R is finitely generated over a commutative ring, then R is isomorphic to a subquotient of $M_n(A)$ for some $n \in \mathbb{N}$.

Cor 2.9 (1) $M_n(A)$ satisfies S_m for all $m \geq n^2+1$

(2) If ${}_AM$ is a f.g. module with n generators, $\mathrm{End}({}_AM)$ is a PI ring

(3) If a ring R is f.g. over a commutative subring, then R is a PI ring

(4) If $A=K$ is a field, R a K-algebra, $n \in \mathbb{N}$, and every element of R is algebraic of degree $\leq n$ over A, then R is a PI-algebra.

__Proof:__ (1) $Z(M_n(A)) = A$ and $M_n(A)$ is generated by $\{e_{i,j} \mid i,j \in [1,n]\}$. Thm 28. implies the claim.

(2) $\exists$ A-module epi: $\pi: {}_AA^n \longrightarrow {}_AM$

Let $S := \{ \varphi \in \underset{\substack{\cong\\ M_n(A)}}{\mathrm{End}({}_AA^n)} \mid \varphi(\ker\pi) \subseteq \ker\pi \} \subseteq \mathrm{End}({}_AA^n)$ (subring)

Every $\varphi \in S$ induces a $\varphi' \in \mathrm{End}({}_AM)$ s.t. $\forall x \in {}_AA^n$: $\varphi'(\pi(x)) = \pi(\varphi(x))$

[$\pi\circ\varphi: {}_AA^n \to {}_AM$ has $\ker\pi \subseteq \ker \pi\circ\varphi$, and hence induces $\varphi': {}_AM \to {}_AM$.]

The map $S \to \mathrm{End}({}_AM)$, $\varphi \mapsto \varphi'$ is a surjective ring hom.

S PI $\Rightarrow$ $\mathrm{End}({}_AM)$ PI.

(3) $R \cong \mathrm{End}(R_R) \subseteq \mathrm{End}(R_A)$, thus, by (2), R is PI.

(4) Thm 28

$\square$

3. The Amitsur-Levitzky Theorem

In this section we will prove

Thm 3.1 (Amitsur-Levitzky) (1) S_{2n} is a PI for $M_n(A)$,
(2) $M_n(A)$ does not satisfy a monic PI of degree $\leq 2n-1$.

Recall: If K is a field, V_K a vector space

$\Lambda(V) = T(V)/\langle\{v\otimes v \mid v\in V\}\rangle$ denotes the exterior (or Grassmann) algebra on V. Its multiplication is denoted by „$\wedge$"
and $\Lambda(V)$ is graded:

$$\Lambda(V) = \bigoplus_{k=0}^{\infty} \Lambda^k(V), \qquad \Lambda^k(V) = {}_K\langle\{v_1\wedge\ldots\wedge v_k \mid v_1,\ldots,v_k\in V\}\rangle$$

If B is a K-basis for V, then

$$\Lambda(V) \cong K\langle B\rangle/\langle\{bb'+b'b,\ b^2 \mid b,b'\in B\}\rangle$$

Lemma 3.2 Let K be a field, V_K a vector space with basis $\{v_1,\ldots,v_n\}$ or countable basis $\{v_1, v_2, \ldots\}$, $n=\dim V$

(a) $M = \{v_{i_1}\wedge\ldots\wedge v_{i_k} \mid k\in\mathbb{N}_0,\ i_1<i_2<\ldots<i_k,\ k\leq n\}$ form a K-basis for $\Lambda(V)$.
If $\dim V = n<\infty$ then $\dim\Lambda(V) = 2^n$.

(b) The monomials of even degree in (a) are central in $\Lambda(V)$.

(c) $f(x,y,z) = [[x,y],z] = (xy-yx)z - z(xy-yx)$ is a PI for $\Lambda(V)$.

Proof: (a) [Sketch] $\Lambda^k(V)=0$ for $k>n$. For $k\leq n$, we show that $M_0=\{v_{i_1}\wedge\ldots\wedge v_{i_k} \mid i_1<\ldots<i_k\}$ is a K-basis of $\Lambda^k(V)$.
Clearly $\Lambda^k(V) = {}_K\langle M_0\rangle$.
For all i, let $f_i\in \mathrm{Hom}_K(V,K)$ be such that $f_i(v_i)=1$, $f_i(v_j)=0$ for $j\neq i$.
For $i_1<\ldots<i_k$ the map $\varphi_{i_1,\ldots,i_k}: V^k \longrightarrow K$, $(w_1,\ldots,w_k)\longmapsto \det([f_{i_s}(w_t)]_{s,t\in[1,n]})$
is multilinear and alternating.

Hence it induces a K-hom $\overline{\varphi}_{i_1,\dots,i_k}: \Lambda^k(V) \longrightarrow K$ (Check!).

Let $j_1 < \dots < j_k$, then

$$\overline{\varphi}_{i_1,\dots,i_k}(v_{j_1}\wedge\dots\wedge v_{j_k}) = \det([f_{i_s}(v_{j_t})]) = \begin{cases} 1 & \text{if } (i_1,\dots,i_k) = (j_1,\dots,j_k) \\ 0 & \text{if } (i_1,\dots,i_k) \neq (j_1,\dots,j_k) \end{cases}$$

$\Longrightarrow$ the elements of M_0 are linearly independent.

(b) Let $i_1 < i_2 < \dots < i_{2k}$. It suffices to show that $v_{i_1}\wedge\dots\wedge v_{i_{2k}}$ commutes with every element v_j, $j \geq 1$, but this is clear.

(c) Since $f(x,y,z)$ is multilinear, it suffices to check that it vanishes on the basis elements (by Lemma 2.3).
If u, v are monomials, and one of u, v has even degree, then $[u,v] = 0$ by (b).
If u, v have odd degree, then uv has even degree (could be 0).
Hence $uv - vu$ is central by (b), and $[[u,v],w] = 0$ $\forall w$.

$\square$

<u>Def. 3.3</u> Let K be a field, $K[t_1,\dots,t_n]$ a commutative polynomial ring.

(1) $f \in K[t_1,\dots,t_n]$ is <u>symmetric</u> if $f(t_{\sigma(1)},\dots,t_{\sigma(n)}) = f(t_1,\dots,t_n)$ for all $\sigma \in \mathfrak{S}_n$.
$K[t_1,\dots,t_n]^{\mathfrak{S}_n}$ denotes the ring of all symmetric polynomials.

(2) $\forall k \in [1,n]$: $\quad e_k(t_1,\dots,t_n) := \sum_{i_1<\dots<i_k} t_{i_1}\cdots t_{i_k}$ is the <u>k-th elementary symmetric polynomial</u>

$p_k(t_1,\dots,t_n) = t_1^k + \dots + t_n^k$ is the <u>k-th power symmetric polynomial</u>

~~Thm 3.4 (Newton's Identities)~~ $\cancel{\boxed{k e_k = \sum_{i=1}^{k} (-1)^k e_{k-i} p_i}}$

Thm 3.4 Let K be a field, char $K=0$, $K[t_1,\dots,t_n]$ a commutative polynomial ring.

(1) $\forall k\in[1,n]$: $$ke_k=\sum_{i=1}^{k}(-1)^k e_{k-i}p_i$$ (Newton's Identities)

(2) $\left(K[t_1,\dots,t_n]^{\mathfrak{S}_n}=\right) K[e_1,\dots,e_n]=K[p_1,\dots,p_n]$.

(3) $\langle e_1,\dots,e_n\rangle_{K[t_1,\dots,t_n]}=\langle p_1,\dots,p_n\rangle_{K[t_1,\dots,t_n]}$

Proof: Omitted.

Lemma 3.5 Let K be a field, char $K=0$, R a commutative K-algebra, and $B\in M_n(R)$.
If $\forall i\in[1,n]$: $\operatorname{tr}(B^i)=0$, then $B^n=0$.

Proof: ~~Let $\overline{K}$ denote the algebraic closure of K, $\lambda_1,\dots,\lambda_n\in\overline{K}$ the eigenvalues of B (repeated according to multiplicity).
The characteristic polynomial of B is~~
$$\cancel{T^n-e_1(\lambda_1,\dots,\lambda_n)T^{n-1}\pm\dots+(-1)^n e_n(\lambda_1,\dots,\lambda_n)\in K[T].}$$
~~By Thm 3.4, the $e_i(\lambda_1,\dots,\lambda_n)$ can be expressed in terms of $p_i(\lambda_1,\dots,\lambda_n)$. Considering the Jordan normal form of B (over $\overline{K}$), we see $\operatorname{tr}(B^i)=p_i(\lambda_1,\dots,\lambda_n)$.
Thus $p_i(\lambda_1,\dots,\lambda_n)=0$ $\forall i\in[1,n]$, and hence also $e_i(\lambda_1,\dots,\lambda_n)=0$.
Thus $B^n=0$.~~ $\square$

Remark: If char $K=p>0$, $I_p\in M_p(K)$, then $\operatorname{tr}(I^i)=\operatorname{tr}(I)=0$, but $I^p\neq 0$.

Lemma 3.6 If $B_1,\dots,B_{2r}\in M_n(A)$, $(r\in\mathbb{N}_0)$ then $\operatorname{tr}(s_{2r}(B_1,\dots,B_{2r}))=0$.

Proof: Let $C:=\langle\underbrace{(1\,2\,\dots\,2r)}_{=:\pi}\rangle\leq S_{2r}$. Let $\sigma\in\mathfrak{S}_{2r}$.
Then
$$\operatorname{tr}\left(\sum_{\tau\in C}\operatorname{sgn}(\tau\sigma)B_{\tau\sigma(1)}\cdots B_{\tau\sigma(2r)}\right)=\sum_{i=0}^{2r-1}(-1)^i\operatorname{sgn}(\sigma)\underbrace{\operatorname{tr}(B_{\pi^i\sigma(1)}\cdots B_{\pi^i\sigma(2r)})}_{\operatorname{tr}(B_{\sigma(1)}\cdots B_{\sigma(2r)})}=0.$$
Partitioning $\mathfrak{S}_{2r}$ into cosets of C, we see $\operatorname{tr}(s_{2r}(B_1,\dots,B_{2r}))=0$. $\square$

Proof of Lemma 3.5

Claim: $\exists\, P_1, \dots, P_{n-1} \in \langle t_1, \dots, t_n \rangle \trianglelefteq \mathbb{Q}[t_1, \dots, t_n]$: $\quad B^n = \sum_{i=0}^{n-1} P_i(\mathrm{tr}(B), \dots, \mathrm{tr}(B^n))\, B^i$.

Proof: Suppose first that $R = L$ is a field, $\bar{L}$ its algebraic closure. The characteristic polynomial of B is

$$T^n - e_1(\lambda_1, \dots, \lambda_n) T^{n-1} \pm \dots \pm (-1)^n e_n(\lambda_1, \dots, \lambda_n) \in L[T], \qquad (*)$$

where $\lambda_1, \dots, \lambda_n \in \bar{L}$ are the eigenvalues of B, repeated according to multiplicity.
By Thm 3.4, the e_i can be expressed in terms of polynomial expressions without constant coefficients in $p_1, \dots, p_n$:

$e_i = P_i(p_1, \dots, p_n)$ for some $P_i \in \langle t_1, \dots, t_n \rangle \trianglelefteq \mathbb{Q}[t_1, \dots, t_n]$.

Considering the Jordan normal form of B (over $\bar{L}$), we see

$\mathrm{tr}(B^i) = p_i(\lambda_1, \dots, \lambda_n)$,

Hence $(*)$ implies the claim in this case.
Suppose now R is an arbitrary commutative K-algebra. Then $\exists$ epi: $\pi: K[\underline{T}] \to R$, which induces $\tilde{\pi}: M_n(K[\underline{T}]) \twoheadrightarrow M_n(R)$, $(a_{i,j})_{i,j \in [1,n]} \mapsto (\pi(a_{i,j}))_{i,j \in [1,n]}$.
Let $B_0 \in M_n(K[\underline{T}])$ with $\tilde{\pi}(B_0) = B$.
Applying what we already showed to $K(\underline{T})$, we see

$$B_0^n = \sum_{i=0}^{n-1} P_i(\mathrm{tr}(B_0), \dots, \mathrm{tr}(B_0^n))\, B_0^i.$$

Noting that $\mathrm{tr}(\tilde{\pi}(B_0)) = \pi(\mathrm{tr}(B_0))$, the claim follows. $\square$

Proof of Thm 3.1

(2) If $M_n(A)$ satisfies a monic PI of degree $\leq 2n-1$, then it satisfies a multilinear monic PI of degree $\leq 2n-1$ by Thm 2.1.
Thus $M_n(A)$ satisfies a PI of the form

$$f(x_1,\dots,x_m) = x_1\cdots x_m + \sum_{\substack{\sigma\in\mathfrak{S}_m\\ \sigma\neq 1}} a_\sigma x_{\sigma(1)}\cdots x_{\sigma(m)}, \quad m\leq 2n-1.$$

Let $e_{i,j}\in M_n(A)$ be the matrix with 1 in the (i,j)-the entry, and 0 everywhere else. Then

$$e_{1,1}\cdot e_{1,2}\cdot e_{2,2}\cdot e_{2,3}\cdot e_{3,3}\cdot e_{3,4}\cdots e_{n-1,n}\, e_{n,n} = e_{1,n}\neq 0$$

is a nonzero product of $2n-1$ matrix units.
The product of any non-trivial permutation of the first m factors is 0. Thus, $f(e_{1,1}, e_{1,2},\dots, e_{n-1,n}, e_{n,n})\neq 0$.

(1) (Rosset)
It suffices to prove that S_{2n} is a PI for $M_n(K)$ with K a field of characteristic 0: Then the same holds for $M_n(\mathbb{Z})$ and hence for $M_n(\mathbb{Z})[T]\cong M_n(\mathbb{Z}[T])$, where T is any set of central indeterminates.
Since $M_n(A)$ is a homomorphic image of $M_n(\mathbb{Z}[T])$ for some set of indeterminates T, S_{2n} is a PI for $M_n(A)$.

Let $U_1,\dots,U_{2n}\in M_n(K)$. Let $V=\langle v_1,\dots,v_{2n}\rangle_K$ be a $2n$-dimensional K-vector space, and let $D\subseteq Z(\Lambda(V))$ be the subring generated by

$$\{v_{i_1}\wedge\dots\wedge v_{i_{2r}} \mid 1\leq i_1<i_2<\dots<i_{2r}\leq 2n\} \quad \text{(monomials of even degree)}.$$

$$C := U_1 v_1 + \dots + U_{2n} v_{2n} \in M_n(\Lambda(V))$$

$$B := C^2 = \sum_{\substack{1\leq i,j\leq 2n\\ i\neq j}} U_iU_j\, v_i\wedge v_j = \sum_{1\leq i<j\leq 2n} (U_iU_j - U_jU_i)\, v_i\wedge v_j$$

$\Longrightarrow$ $B \in M_n(D)$ with D commutative.

$\forall k \in [1,n]$:

$$B^k = C^{2k} = \sum_{\substack{i_1,\dots,i_{2k} \in [1,2n] \\ i_j \neq i_h \text{ for } j \neq h}} U_{i_1} \cdots U_{i_{2k}} V_{i_1} \wedge \dots \wedge V_{i_{2k}}$$

$$V_{\sigma(i_1)} \wedge \dots \wedge V_{\sigma(i_{2k})} = \operatorname{sgn}(\sigma)\, V_{i_1} \wedge \dots \wedge V_{i_{2k}}$$

$$= \sum_{1 \leq i_1 < i_2 < \dots < i_{2k} \leq 2n} S_{2k}(U_{i_1}, \dots, U_{i_{2k}})\, V_{i_1} \wedge \dots \wedge V_{i_{2k}}$$

$\overset{\text{Lemma 3.6.}}{\Longrightarrow}$ $\forall k \in [1,n]$: $\operatorname{tr}(B^k) = 0$

$\overset{\text{Lemma 3.5.}}{\underset{B \in M_n(D)}{\Longrightarrow}}$ $0 = B^n = S_{2n}(U_1, \dots, U_{2n})\, V_1 \wedge \dots \wedge V_{2n}$

$\overset{V_1 \wedge \dots \wedge V_{2n} \text{ is a basis elt in } \Lambda(V)}{\Longrightarrow}$ $S_{2n}(U_1, \dots, U_{2n}) = 0$.

$\square$

4. Central Polynomials

Def 4.1 Let R be an A-algebra. $f(x_1,\dots,x_m) \in A\langle X\rangle$ is a *central polynomial* for R if

(i) $\forall r_1,\dots,r_m \in R$: $f(r_1,\dots,r_m) \in Z(R)$,

(ii) f is not a PI for R,

(iii) the constant term of f equals 0.

Here (i) can be replaced by

(i') $[f, x_{m+1}]$ is a PI for R

Examples: Let K be a field

(1) x is a central polynomial for every commutative ring $R \neq 0$.

(2) If V_K is a vector space, $[x,y]$ is a central poly for $\Lambda(V)$

(3) $[x,y]^2$ is a central polynomial for $M_2(K)$ (Hall identity)

(4) $T_n(K)$ ($n \geq 2$, $|K| = \infty$) has no central polynomials

Remarks (1) If R has a central polynomial, then R satisfies a PI (by (i'))

(2) If R & S are A-algebras satisfying the same [multilinear] identities, then they have the same [multilinear] central polynomials. (again by (i'))

(3) If $R = \langle M\rangle_{Z(R)}$ for some set M, f is multilinear, and $\forall m_1,\dots,m_n \in M$: $f(m_1,\dots,m_n) \in Z(R)$, then (i) holds.

We will construct a multilinear central polynomial for $M_n(K)$, following a construction of Razmyslov (as presented in L.H. Rowen, Graded Algebras: A non commutative view).

Throughout this section, let K be a field

Lemma 4.2 Let $n \geq 1, \forall i,j \in [1,n]: e_{i,j} \in M_n(K)$ the elementary polynomials.

(1) $\forall i,j \in [1,n], i \neq j: \quad e_{i,j} = [e_{i,j}, e_{j,j}]$ and $e_{i,i} - e_{1,1} = [e_{i,1}, e_{1,i}]$

(2) $U = \{B \in M_n(K) \mid Tr(B) = 0\}$ is an n^2-1-dimensional vector space spanned by commutators.

Proof: (1) ✓ (2) $\{e_{i,j} \mid i,j \in [1,n]\} \cup \{e_{i,i} - e_{1,1} \mid i \in [2,n]\}$ is a set of n^2-1 linearly independent elements of U, and $\dim(U) = n^2-1$. □

Def 4.3 A *k-weak identity* ($k \in \mathbb{N}$) of $M_n(K)$ is a polynomial $f(x_1,\dots,x_m) \in K\langle X \rangle$ s.t. $f(U_1,\dots,U_k,\dots,U_m) = 0$ whenever $U_1,\dots,U_m \in M_n(K)$ with $tr(U_1) = \dots = tr(U_k) = 0$

Lemma 4.4 $f(x_1,\dots,x_m) \in K\langle X \rangle$, linear in x_k is a k-weak identity of $M_n(K)$ ($k \leq m$) if and only if $g(x_1,\dots,x_{m+1}) = f(x_1,\dots,x_{k-1},[x_k,x_{m+1}],x_{k+1},\dots,x_m)$ is a $(k-1)$-weak identity for $M_n(K)$

Proof: „$\Rightarrow$": Since $tr([U_k,U_{m+1}]) = 0$ for $U_k, U_{m+1} \in M_n(K)$.

„$\Leftarrow$": Let $U_1,\dots,U_m \in M_n(K)$ with $tr(U_1) = \dots = tr(U_k) = 0$.

By lemma 4.2, $\exists \lambda_1,\dots,\lambda_e \in K; V_1, W_e' \in M_n(K)$: $U_k = \sum_{j=1}^{e} \lambda_j [V_j, W_j]$.

f linear in $x_k \rightarrow$

$$f(U_1,\dots,U_k,\dots,U_m) = \sum_{j=1}^{e} \lambda_j f(U_1,\dots,\overset{k}{\downarrow}{[V_j,W_j]},\dots,U_m) = \sum_{j=1}^{e} \lambda_j g(U_1,\dots,\overset{k}{\downarrow}{V_j},\dots,U_m,W_j)$$

$= 0$.

(*) $\rightarrow$ A polynomial $f(x_1,\dots,x_m) \in K\langle X \rangle$ is t-alternating ($t \leq m$), if it is alternating in $x_1,\dots,x_t$ □

Def 4.5: For $t \in \mathbb{N}_0$, the *t-th Capelli polynomial* is

$$c_t = \sum_{\sigma \in S_t} sgn(\sigma)\, x_{\sigma(1)} Y_1 x_{\sigma(2)} Y_2 \cdots x_{\sigma(t)} Y_t$$

Proposition 4.6

(1) c_t is multilinear (in $x_1, \dots, x_t, y_1, \dots, y_t$) and t-alternating (alternating in $x_1, \dots, x_t$).

(2) c_{n^2} is an n^2-weak identity for $M_n(K)$.

(3) $e_{i,j} \in c_{n^2}(M_n(K))$ for all $i,j \in [1,n]$. In particular, $c_{n^2}(M_n(K)) \neq 0$.

(4) $h_n = c_{n^2}(x, [z_2, z_2'], \dots, [z_{n^2}, z_{n^2}']; y_1, \dots, y_{n^2})$ is a multilinear 1-weak identity for $M_n(K)$ that is not an identity.

Proof: (1) ✓

(2) Since c_{n^2} is multilinear and alternating in $x_1, \dots, x_{n^2}$, whereas $\{B \in M_n(K) \mid tr(B) = 0\}$ is generated by n^2-1 elts (as K-vector space) (argue with pigeonhole principle, as in Lemma 2.6)

(3) Let the n^2 elementary matrices be ordered in an arbitrary way:

$$e_{p_1,q_1}, \; e_{p_2,q_2}, \; \dots, \; e_{p_{n^2},q_{n^2}}.$$

We will substitute those for the $x_1, \dots, x_{n^2}$, while we substitute

$$e_{q_1,p_2}, \; e_{q_2,p_3}, \dots, \; e_{q_i, p_{i+1}}, \dots, e_{q_{n^2-2}, p_{n^2-1}}, \; e_{q_{n^2-1}, p_{n^2}}, \; e_{q_{n^2}, q_1}$$ for the y_i.

In any monomial of c_{n^2} not evaluating to 0, the $y_1, \dots, y_{n^2}$ then force $x_{\sigma(n^2)} = e_{p_{n^2}, q_{n^2}}$, $x_{\sigma(n^2-1)} = e_{p_{n^2-1}, q_{n^2-1}}, \dots$, $x_{\sigma(2)} = e_{p_2,q_2}$ and finally $x_{\sigma(1)} = e_{p_1,q_1}$ (since it is the only one left over).

$$\Longrightarrow c_{2t}(e_{p_1q_1}, \dots, e_{p_{n^2}, q_{n^2}}; e_{q_1,p_2}, \dots, e_{q_{n^2}, q_1}) = e_{p_1q_1} e_{q_1p_2} \cdots e_{q_{n^2}, q_1} = e_{p_1,q_1}$$

Since the choice of e_{p_1,q_1} was arbitrary / by symmetry, $e_{i,j} \in c_{n^2}(M_2(K))$ for all $i,j \in [1,n]$.

(4) Combining (2) and Lemma 4.4 (applied (n^2-1)-times), we see that h_n is a 1-weak identity.

We can write $e_{i,i} - e_{1,1} = [U_i, U_i']$ and $(e_{i,j})_{\substack{i,j \in [1,n] \\ i \neq j}} = ([U_k, U_k'])_{k \in [n+1, n^2]}$ with suitable $U_i, U_i' \in M_n(K)$ (Lemma 4.2)

Substitution $z_i = U_i$, $z_i' = U_i'$, $x = e_{1,1}$, we get

$$h_n(\dots) = c_{n^2}(e_{1,1}, e_{2,2} - e_{1,1}, e_{3,3} - e_{1,1}, \dots, e_{n,n} - e_{1,1}, e_{1,2}, \dots, e_{n-1,n}; y_1, \dots, y_{n^2})$$

$$\underset{\substack{\uparrow \\ n^2\text{-alternating multilin.}}}{=} c_{n^2}(e_{1,1}, e_{2,2}, e_{3,3}, \dots, e_{n,n}, e_{1,2}, \dots, e_{n-1,n}; y_1, \dots, y_{n^2}),$$

and a suitable substitution for $Y_{1}, \ldots, Y_{n^2}$ (as in (3)) shows $h_n(\_) = e_{1,n}$.

$\square$

Def 4.7 Let R be a ring, $k \in \mathbb{N}_0$, $\forall i \in [1,k]: a_i, b_i \in R$.

An expression $f = \sum_{i=1}^{k} a_i x b_i$ is

- a *linear generalized identity* (LGI) for R if $\forall r \in R: f(r) := \sum_{i=1}^{k} a_i r b_i = 0$
- *linear generalized central* (LGC) for R if $\forall r \in R: f(r) \in Z(R)$

We set $f^* = \sum_{i=1}^{k} b_i x a_i$ (Resmyslov transform)

(A „generalized polynomial" is a noncommutative polynomial with coefficients from R interspersed throughout the indeterminates x_i. We only need the linear special case.)

NOTE: $M_n(K) \cong M_n(K)^{op}$ by means of the transpose $B \mapsto B^T$.

Lemma 4.8 Let R be a ring, $k \in \mathbb{N}_0$, $\forall i \in [1,k]: a_i, b_i \in R$.

(1) $f = \sum_{i=1}^{k} a_i x b_i$ is a LGI for $R \iff f^* = \sum_{i=1}^{k} b_i x a_i$ is a LGI for R^{op}.

(2) Let $R = M_n(F)$.

Then $f^* = \sum_{i=1}^{k} b_i x a_i$ is LGC (for R)

$\iff [\forall r \in R: \operatorname{Tr}(r) = 0 \implies f(r) = 0]$

Proof: (1) $\forall r \in R: 0 = \sum_{i=1}^{k} a_i r b_i \iff 0 = \sum_{i=1}^{k} b_i \cdot_{op} r \cdot_{op} a_i$

(2) Due to lemma 4.2, the 2nd condition is equivalent to

$\forall r, s \in R: \quad 0 = f([r,s]) = \sum_{i=1}^{k} a_i r s b_i - a_i s r b_i$.

This is equivalent to $g := \sum_{i=1}^{k} a_i x s b_i - a_i s x b_i$ being an LGI for R.

By (1), g is a LGI for R iff $g^* = \sum_{i=1}^{k} s b_i x a_i - b_i x a_i s = [s, f^*]$ is an LGI for $R^{op} \cong R$. Finally, $[s, f^*]$ being an LGI (for R) is equivalent to f^* being LGC for R.

$\square$

<u>Theorem 4.9</u> (Razmyslov) There is a bijective correspondence between multilinear central polynomials of $M_n(K)$ and multilinear 1-weak identities that are not identities and have constant term 0.
It is given by

$$\sum_{i=1}^{k} f_i(x_2,\dots,x_m)\,x_1\,g_i(x_2,\dots,x_m) \longmapsto \sum_{i=1}^{k} g_i(x_2,\dots,x_m)\,x_1\,f_i(x_2,\dots,x_m)$$

<u>Proof:</u> Let $f \in K\langle x_1,\dots,x_m\rangle$ be multilinear with constant term 0, and suppose that f is not a PI for $M_n(K)$.
Write $f = \sum_{i=1}^{k} f_i(x_2,\dots,x_m)\,x_1\,g_i(x_2,\dots,x_m)$ with $f_i, g_i \in K\langle x_2,\dots,x_m\rangle$

For given $B_2,\dots, B_m \in M_n(K)$, let $a_i := f_i(B_2,\dots,B_m)$, $b_i = g_i(B_2,\dots,B_m)$.
Hence $h := \sum_{i=1}^{k} b_i x_1 a_i$ is

LGC $\Leftrightarrow$ $\forall B \in M_n(K)$: $\operatorname{tr}(B)=0 \Rightarrow h(B)=0$.
Since this is true for any choice of matrices, f is a central polynomial if and only if $\sum_{i=1}^{k} g_i x_1 f_i$ is a 1-weak identity. □

<u>Corollary 4.10</u> For any $n \in \mathbb{N}$, there is a multilinear central polynomial g_n for $M_n(K)$ for all fields K.

<u>Proof:</u> Combining Thm 4.9 with Prop 4.6 (4), we see that the 1-weak identity $C_{n^2}(X, [Z_1,Z_1'],\dots,[Z_{n^2},Z_{n^2}'] ; Y_1,\dots,Y_{n^2})$ gives rise to a multilinear central polynomial. □

<u>Remark:</u> One can also construct an <u>alternating</u> multilinear central polynomial.

<u>Lemma 4.11</u> Any polynomial $f(x_1,\dots,x_m) \in A\langle X\rangle$ that is an identity or a central polynomial for $M_n(A)$ is an identity for $M_\ell(A)$, $\ell < n$.
In particular, g_n is an identity for $M_\ell(A)$, $\ell < n$.

Proof: Embed $M_\ell(A)$ into the upper left corner of $M_n(A)$ as ring without 1. (By placing 0's everywhere else), $\varepsilon: M_\ell(A) \hookrightarrow M_n(A)$

We can assume that the constant term of f is 0 (since accordingly $f(0,\dots,0)\cdot 1_4 = 0_4$)

Then, $\forall B_1,\dots,B_m \in M_\ell(A)$: $\varepsilon(f(B_1,\dots,B_m)) = f(\varepsilon(B_1),\dots,\varepsilon(B_m))$

If f is a PI for $M_n(A)$, then $f(B_1,\dots,B_m)=0$.

If f is a central polynomial for $M_n(A)$, then

$B := f(\varepsilon(B_1),\dots,\varepsilon(B_m)) = \begin{bmatrix} c & & 0 \\ & \ddots & \\ 0 & & c \end{bmatrix}$ for $c \in A$, and then $c = 0$ since $B \in \varepsilon(M_\ell(A))$. $\square$

Def. 4.12: A ring R has PI class $n (\in \mathbb{N})$ if

(i) R satisfies all multilinear identities of $M_n(\mathbb{Z})$

(ii) g_n (as constructed in Cor. 4.10) is a central polynomial for R.

Careful: The definition of PI class varies quite a bit in different texts!

Prop 4.13 $M_n(A)$ has PI class n, for any commutative ring A ($n \in \mathbb{N}$)

Proof: $M_n(A) \cong A \otimes_{\mathbb{Z}} M_n(\mathbb{Z})$, hence $M_n(A)$ is a central extension of the image of $M_n(\mathbb{Z})$ in $M_n(A)$. Thus, $M_n(A)$ satisfies any multilinear identity of $M_n(\mathbb{Z})$ (by Cor 2.4.). In particular, it satisfies $[z, g_n]$ and hence g_n is either an identity or a central polynomial.

If g_n is an identity for $M_n(A)$, then it is an identity for $M_n(A/P)$ with $P \triangleleft A$ maximal, contradicting Cor 4.10. $\square$

5 Kaplansky's Theorem

Let R be a ring. M_R is *simple* if $M_R \neq 0$ and $0, M$ are the only submodules of M. M_R is *faithful* if $0 = \mathrm{ann}(M) = \{r \in R \mid \forall m \in M: mr = 0\}$.
A ring R is simple if $0 \neq R$ and $0, R$ are the only ideals of R.

Def. 5.1 A ring R is *(right) primitive* if there exists a faithful simple left R-module.

Remarks (1) There exist right primitive rings that are not left primitive.

(2) A commutative ring is primitive if and only if it is a field.

[„$\Leftarrow$" ✓, „$\Rightarrow$": Let M_R be faithful simple. Then $M \cong R/m$ for a maximal ideal $m \trianglelefteq R$. Since $Mm = 0$, $m = 0$. Hence R is a field.]

(3) Division rings, and more generally simple rings, are primitive.

(4) R primitive $\Leftrightarrow$ $M_n(R)$ primitive [Morita equivalence]

☆

Def 5.4: Let R be a ring, D a division ring, ${}_DM_R$ a bimodule which is faithful as R-module. [Then $R \hookrightarrow \mathrm{End}_D(M)$, $r \mapsto (m \mapsto mr)$]
We say that R is a *dense subring* of $\mathrm{End}_D(M)$ if whenever $m_1, \dots, m_k \in M$ are D-linearly independent and $n_1, \dots, n_k \in M$, there exists an $r \in R$ such that $m_i = n_i r$ for all $i \in [1, k]$.

Note: If $0 \neq R$ is a dense subring of $\mathrm{End}_D(M)$, $0 \neq m \in M$, $m' \in M$ then there exists $r \in R$ such that $mr = m'$. Thus $mR = M$, and M is simple.

Theorem 5.5 (Jacobson Density Theorem) Let R be a primitive ring, M_R a simple faithful R-module, and $D = \mathrm{End}(M_R)$. Then M is a (D,R)-bimodule and R is a dense subring of $\mathrm{End}_D(M)$.

Recall: ~~If~~ R is a simple ring, then $Z(R)$ is a field.
[$\lambda \in Z(R) \Rightarrow R\lambda = R \Rightarrow \lambda \in R^\times$. But then also $\lambda^{-1} \in Z(R)$]
(Without proof)

Def. 5.2 Let K be a field. A central simple K-algebra (CSA/K) is a K-algebra A such that $K = Z(A)$, A is a simple ring, and $\dim_K A < \infty$.
Equivalently, by Artin-Wedderburn, $A \cong M_n(D)$ for $n \in \mathbb{N}$ and a division ring D with $Z(D) = K$ and $\dim_K D < \infty$.

Thm 5.3 (Kaplansky) A primitive PI ring is a central simple algebra over its center.

<u>Lemma</u> 5.6 Let R be a primitive ring, M_R a faithful simple module, and $D = End(M_R)$. Then either

(i) $\dim_D M = n$ and $R \cong M_n(D)$ for some $n < \infty$; or

(ii) for each $n \in \mathbb{N}$ there is a subring $R_n \subseteq R$ and a surjective ring homomorphism $R_n \twoheadrightarrow M_n(D)$.

<u>Proof</u>: Suppose first $\dim_D M = n < \infty$. Then $End({}_DM) \cong M_n(D)$ and R is dense in $End({}_DM)$. The only dense subring of $M_n(D)$ is $M_n(D)$ itself, hence $R \cong M_n(D)$ and (i) holds.

Suppose now $\dim_D M = \infty$ and let $n \in \mathbb{N}$.

Let $e_1, \dots, e_n \in M$ be D-linearly independent and

$${}_DM_n = {}_D\langle e_1, \dots, e_n \rangle \subseteq {}_DM.$$

Set $R_n = \{ r \in R \mid rM_n \subseteq M_n \}$. Then there is a ring hom $R_n \longrightarrow End(M_n) \cong M_n(D)$. By the Jacobson density theorem, it is surjective.

$\square$

<u>Recall</u>: If R is a simple ring, then $Z(R)$ is a field

[$\lambda \in Z(R) \Rightarrow R\lambda = R \Rightarrow \lambda \in R^\times$. Then also $\lambda^{-1} \in Z(R)$]

Prop 5.7 Let R be a simple ring, $K = Z(R)$, and S a simple K-algebra. Then $R \otimes_K S$ is a simple K-algebra.

Proof: Let $0 \neq I \trianglelefteq R \otimes S$.

Choose $0 \neq x = \sum_{i=1}^{n} r_i \otimes s_i \in I$ with $r_i \in R$, $s_i \in R$ and $\underline{n \text{ minimal}}$.

Then $s_1, \dots, s_n$ are K-linearly independent.

Also $r_1 \neq 0$ and hence $R r_1 R = R$.

If $a_1, \dots, a_m, b_1, \dots, b_m \in R$ are such that $\sum_{j=1}^{m} a_j r_1 b_j = 1$,

$$I \ni \sum_{j=1}^{m} (a_j \otimes 1) x (1 \otimes b_j) = 1 \otimes s_1 + \sum_{i=1}^{n} \Big(\sum_{j=1}^{m} a_j r_i b_j \Big) \otimes s_i \neq 0$$

↑ by linear indep of s_i

Hence, we may without restriction assume $r_1 = 1$.

For all $r \in R$,

$$I \ni (r \otimes 1) x - x (r \otimes 1) = \sum_{i=2}^{n} (r r_i - r_i r) \otimes s_i$$

By minimality of n, $\sum_{i=2}^{n} (r r_i - r_i r) \otimes s_i = 0$. By K-linear independence of the s_i, we find $\forall i \in [2,n]: \ r r_i = r_i r$.

Thus, $\forall i \in [2,n]: \ r_i \in Z(R) = K$.

$$\Longrightarrow \quad x = 1 \otimes s_1 + \sum_{i=2}^{n} r_i \otimes s_i = 1 \otimes \underbrace{\Big(s_1 + \sum_{i=2}^{n} r_i s_i \Big)}_{\in S} = 1 \otimes y \quad \text{with } y \in S.$$

Since $0 \neq x$, also $y \neq 0$.

S simple $\Longrightarrow$ $SyS = S$. $\Longrightarrow$ $1 \otimes 1 \in I \Rightarrow I = R \otimes_K S$

$\square$

Def 5.8 A subfield L of a ring R is called a *maximal subfield* if $L \subset L' \subset R$ with a subfield L' of R implies $L = L'$.

If R contains a field, then it contains a maximal subfield by Zorn's Lemma. Even if R is simple, so that $Z(R)$ is a field, $Z(R)$ may not be a maximal subfield!

Lemma 5.9 Let D be a division ring, $K = Z(D)$, and L a maximal subfield of D. Let $R = M_n(D)$ for some $n \in \mathbb{N}$ and M_R a simple module. (Then $D \cong \mathrm{End}(M_R)$)

(1) $S = R \otimes_K L$ is simple and M is a simple right S-module with $\mathrm{End}\, M_S \cong L$.

(2) If $\dim_L M = m < \infty$, then $S \cong M_m(L)$, $\dim_L D = \dim_K L = \frac{m}{n}$ and $\dim_K R = m^2$.

Proof: (1) Prop 5.7 implies that S is simple.
One checks that a right S-module structure on M is given by
$\forall m \in M\ \forall r \in R\ \forall \ell \in L: \quad m(r \otimes \ell) = \ell m r.$ [Construct K-bil. maps $R \times L \to \mathrm{End}(M)$]
Since M_R is simple, clearly so is M_S.

Claim: $\mathrm{End}(M_S) \cong \{x \in D \mid \forall \ell \in L: x\ell = \ell x\} = L$.

Proof of Claim: The second equality follows from the maximality of L.
We have $\alpha: D \xrightarrow{\sim} \mathrm{End}(M_R)$, $d \mapsto \alpha_d$ with $\alpha_d(m) = dm$ for $m \in M$.
Since $\mathrm{End}(M_S) \subseteq \mathrm{End}(M_R)$, $\mathrm{End}(M_S)$ is isomorphic to a subring of D. Let $d \in D$.
$\alpha_d \in \mathrm{End}(M_S) \iff \forall m \in M, r \in R, \ell \in L: \underbrace{\alpha_d(m(r\otimes\ell))}_{= d\ell m r} = \underbrace{\alpha_d(m)(r\otimes\ell)}_{= \ell d m r}$
Clearly, if $d\ell = \ell d$ for all $\ell \in L$, then $\alpha_d \in \mathrm{End}(M_S)$.
Conversely, if $d\ell m = \ell d m$ $\forall \ell \in L, \forall m \in M$, then $d\ell = \ell d$ $\forall \ell \in L$, since ${}_D M$ is faithful

(2) By Lemma 5.6 (i) and (1), $S \cong M_m(L)$ for some $m \in \mathbb{N}$.
Since $S \cong R \otimes_K L$, $\dim_L S = \dim_K R$ (by Prop I.1.14)
Thus $\underline{m^2 = \dim_K R} = n^2 \dim_K D$. $(*)$
Then ${}_L L^m \cong {}_L M \cong {}_L D^n$ implies $\dim_L D = \frac{m}{n}$
And from $(*)$ we conclude $\dim_K L \cdot \underbrace{\dim_L D}_{=\frac{m}{n}} = \frac{m^2}{n^2}$, hence $\dim_K L = \frac{m}{n}$. □

Cor 5.10 Let $R = M_n(D)$ be a central simple algebra, where D is a division ring with center K. Let $L \subset D$ be a maximal subfield.

(1) $D \otimes_K L \cong M_m(L)$ where $m^2 = \dim_K D$,
and $R \otimes_K L \cong M_t(L)$ where $t^2 = m^2 n^2 = \dim_K R$.

(2) R satisfies the same multilinear identities (in $\mathbb{Z}\langle X \rangle$) as $M_t(L)$.

(3) The polynomial g_t (from Cor 4.10) is a central polynomial for R

(4) R has PI class t (with $t^2 = \dim_{Z(R)} R$)

Proof. (1) ✓ by Lemma 5.9.

(2) Let $f \in \mathbb{Z}\langle X \rangle$ be a multilinear identity of $M_t(L)$.
Since $R \cong \{r \otimes 1 \mid r \in R\} \subset M_t(L)$, f is an identity of R.
Conversely, if f is a multilinear identity of R, then it is also an identity of $R \otimes_K L \cong M_n(L)$

(3) By (2) and Cor 4.10 $[g_t, z]$ is an identity for R, but g_t is not.

(4) If $f \in \mathbb{Z}\langle X \rangle$ is a multilinear identity of $M_t(\mathbb{Z})$, then it is an identity of the central extension $M_t(\mathbb{Z}) \otimes_{\mathbb{Z}} L \cong M_t(L)$ of R by (3). Together with (4), this implies that R h... and hence PI class ... □

Proof of Thm 5.3 (Kaplansky's Theorem)

Let R be a primitive PI ring, M_R a faithful simple module, $D = End(M_R)$, $K = Z(D)$.

Let $f \in \mathbb{Z}\langle X \rangle$ be a monic multilinear identity of R of minimal degree, and let $d = \deg f$.

By Lemma 5.6, either $\exists n \in \mathbb{N}$: $\dim_D M = n < \infty$ and $R \cong M_n(D)$,

or $\forall n \in \mathbb{N}$: $\exists$ subring $R_n \subset R$ and a surjective ring hom. $\varphi_n : R_n \twoheadrightarrow M_n(D)$.

In the second case, $M_n(D)$ satisfies f (for all $n \in \mathbb{N}$), and hence so does $M_n(K)$. Thus $d \geq 2n$ for all $n \in \mathbb{N}$ by Thm 5.1(2), a contradiction.

Hence $\exists n \in \mathbb{N}$: $R \cong M_n(D)$, $\dim_D M = n$.

Let $L \subset D$ be a maximal subfield. By Lemma 5.9, M is a simple faithful right $R \otimes_K L$-module with $End(M_{R \otimes_K L}) \cong L$. Moreover, $R \otimes_K L$ is simple (also by Lemma 5.9).

Since R is a PI ring, so is $R \otimes_K L$. Thus $R \otimes_K L$ is a primitive PI ring, and by what we already showed, $\dim_L M < \infty$.

By Lemma 5.9(2), $\dim_K R = (\dim_L M)^2 < \infty$. $\square$

Cor 5.11 If R is a primitive PI ring and d is the minimal degree of a monic PI for R, then $\dim_{Z(R)} R = (\frac{d}{2})^2$.

Remark: The converse to Kaplansky's Thm is trivially true: Every CSA is a primitive PI ring (see Thm 28)

6. Two Theorems on Radicals

We prove two theorems that will be needed for the proof of Posner's Theorem. The proof of the first of these theorems uses an inductive argument that forces us to consider non-unital rings.

Def. 6.1 A rng ("ring") R is a ring that may not have a unit element. R is a domain if $ab=0$ implies $a=0$ or $b=0$ for $a,b\in R$. R is prime if $IJ=0$ implies $I=0$ or $J=0$ for $I,J \triangleleft R$.

In a rng R, the ideal generated by a set M is

$${}_R\langle M\rangle_R := \sum_{m\in M} \mathbb{Z}m + Rm + mR + RmR.$$

Lemma 6.2 Let R be a rng. TFAE:

(a) R is prime.

(b) $\forall$ left ideals $I,J\subset R$: $IJ=0 \Rightarrow I=0$ or $J=0$.

(c) $\forall$ right ——— " ———.

(d) If $a,b\in R$ are s.t. $aRb=0$ and $ab=0$, then $a=0$ or $b=0$.

Proof: We show $(a)\Rightarrow(d)\Rightarrow(b)\Rightarrow(a)$.

$(a)\Rightarrow(d)$: ${}_R\langle a\rangle_R \cdot {}_R\langle b\rangle_R = 0$, hence $a=0$ or $b=0$.

$(d)\Rightarrow(b)$: By contradiction. Suppose $IJ=0$ with $I\neq 0$ and $J\neq 0$.

$\Rightarrow \exists a\in I\setminus\{0\},\ b\in J\setminus\{0\}$: $ab=0$ and $aRb=0$. $\lightning$

$(b)\Rightarrow(a)$: ✓

$\square$

It makes sense to consider PIs of R, as long as they have zero constant coefficient.

We cannot specialize PIs by substituting constants. E.g. $[x_1,x_2]$ is not a consequence of $x_3[x_1,x_2]$.

To be able to consider monic PIs, we still require that the base ring A is unital.

A rng R is an *A-algebra* if it is an A-module and
$\forall r,s \in R\ \forall a \in A$: $\quad a(rs) = (ar)s = r(as)$.

(Any rng is a $\mathbb{Z}$-algebra.)

Thm 2.1 (multilinearization) still holds for rngs.

Lemma 6.3 Let R be a prime rng and L a left ideal of R.
Let $\operatorname{rann}(L) = \{r \in R \mid Lr = 0\}$.

(1) $I := \operatorname{rann}(L) \cap L$ is an ideal of ~~A~~. the rng L.

(2) L/I is a prime rng.

Proof: (1) $L\operatorname{rann}(L) = 0 \Rightarrow LI = 0 \subset I$.

$L(IL) = 0 \Rightarrow IL \subset \operatorname{rann}(L) \cap L = I$.

(2) Let $J, K \trianglelefteq L$ with $JK \subset I$.

$$\underbrace{(LJ + LJR)}_{\trianglelefteq R}\underbrace{(LK + LKR)}_{\trianglelefteq R} \subset LJK + LJKR \subset LI + LIR = 0.$$

$\overset{R \text{ prime}}{\Longrightarrow} LJ + LJR = 0$ or $LK + LKR = 0 \Rightarrow J \subset I$ or $K \subset I$.

$\square$

Def. 6.4 ~~A subset~~ Let R be a rng. A subset $M \subset R$ is *nil* if every element of M is nilpotent.

Thm 6.5 (Levitzki)(*) If R is a prime PI-rng (as A-algebra), then R contains no nonzero ~~ideals~~. nil ideals.

Proof: Since Thm 2.1 (multilinearization) also holds for rngs, R satisfies a monic multilinear identity

$$f = x_1 \cdots x_n + \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma \neq 1}}{}' a_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)} \in A\langle X \rangle.$$

(*): Better: Formulate as in McR Thm 13.2.5; Then we get semiprime as well

We proceed by induction on $n = \deg f$.

<u>$n=1$:</u> Then $f = x_1$ and hence $R = 0$ ✓

<u>$n>1$, $n-1 \to n$:</u> Suppose R contains a nonzero nil ideal N.

Let $0 \neq b \in N$ and $k \geq 2$ with $b^k = 0$, $b^{k-1} \neq 0$.

Setting $a = b^{k-1}$ we have $a \in N$, $a^2 = 0$, and $a \neq 0$.

$L := Ra$ is a left ideal of R.

[<u>Careful</u>: This may not be the left ideal generated by a!]

By Lemma 6.3(2), $S := L/(\mathrm{rann}(L) \cap L)$ is a prime ring.

<u>Claim</u>: S satisfies a monic multilinear identity of degree $n-1$.

<u>Proof of Claim</u>: Let $f(x_1, \dots, x_n) = x_1 g(x_2, \dots, x_n) + h(x_1, \dots, x_n)$

with no monomial of h starting with x_1.

$\forall r_1, \dots, r_n \in R$: $h(ar_1, r_2 a, \dots, r_n a) = 0$ due to $a^2 = 0$.

$\Longrightarrow$ $0 = f(ar_1, r_2 a, \dots, r_n a) = ar_1 g(r_2 a, \dots, r_n a)$

$\Longrightarrow$ $aRg(r_2 a, \dots, r_n a) = 0$

Similarly, replacing ar_1 by a, we see $ag(r_2 a, \dots, r_n a) = 0$.

Since R is prime, $\forall r_2, \dots, r_n \in R$: $g(r_2 a, \dots, r_n a) = 0$ (Lemma 6.2).

$\Longrightarrow$ $g(x_2, \dots, x_n)$ is a monic multilinear identity of degree $n-1$ for L and hence for S.

□ (Claim)

By IH, S has no nonzero nil ideals.

Since $L = Ra \subset N$ is nil $\Longrightarrow$ S is nil.

$\Longrightarrow$ $0 = S = L/(\mathrm{rann}\, L \cap L)$ $\Longrightarrow$ $RaRa = L^2 = 0$ $\xrightarrow{R \text{ prime}}$ $Ra = 0$.

$Ra = 0$ and $a^2 = 0$ implies $a = 0$ since R is prime.

□

Def 6.7 The Jacobson radical, $J(R)$, of a ring R is the intersection of all maximal left ideals of R.

Thm 6.8 Let R be a ring.

(1) $J(R) = \bigcap \{M \subset R \mid M \text{ maximal right ideal}\}$

$= \bigcap \{M \triangleleft R \mid R/M \text{ is left primitive}\}$

$=$ — " ——— right — " —

$= \{r \in R \mid \forall x \in R: 1 - xr \in R^\times\}$

$= \{r \in R \mid \forall x \in R: 1 - rx \in R^\times\}$

(2) $J(R) \triangleleft R$

(3) $J(R)$ contains every nil ideal of R.

Proof: See any ring theory textbook.

Def 6.9 A ring R is

(1) semiprime if $I^2 = 0 \Rightarrow I = 0$ for all $I \triangleleft R$.

(2) semiprimitive if $J(R) = 0$.

Every prime [primitive] ring is semiprime [semiprimitive].
Every semiprimitive ring is semiprime (by Thm 6.8(3))

Prop. 6.10 Let R be a ring and $f = a_n T^n + \dots + a_1 T + a_0 \in R[T]$, where $a_0, \dots, a_n$ commute. Then $f \in R[T]^\times \Leftrightarrow a_0 \in R^\times$ and $a_1, \dots, a_n$ are nilpotent.

Proof: Clearly, $a_0 \in R^\times$ is necessary for $f \in R[T]^\times$. So, wrt assume $a_0 \in R^\times$.
Let $R_0 \subset R$ be the commutative subring generated by $a_0, a_0^{-1}, a_1, \dots, a_n$.
Then $f \in R_0[T]^\times$. Hence $f \in R[T]^\times \Leftrightarrow f \in R_0[T]^\times$.
We may assume wrt. that $R = R_0$ is commutative.

„⇐": $\forall i \in [1,n]$: $a_i T^i$ is nilpotent, hence so is $f_1 := a_1 T^1 + \dots + a_n T^n$.
Let $k \geq 1$ be s.t. $f_1^k = 0$.

Then $$\underbrace{(a_0 - (-f_1))}_{=f} \left(\sum_{i=0}^{k-1} a_0^i (-f_1)^{k-1-i} \right) = a_0^k - (-f_1)^k = a_0^k \in R^\times,$$

hence $f \in R[T]^\times$.

„⇒": Let $P \in \operatorname{Spec}(R)$. Since $\bar{f} \in R/P[T]^\times$, we have $a_i \in P$ for $i \in [1,n]$

$\longrightarrow \forall i \in [1,n]$: $a_i \in \bigcap_{P \in \operatorname{Spec}(R)} P = \sqrt{0} \longrightarrow a_i$ nilpotent.

~~holds for semiprime, but due to book arrangement proof here only for prime~~

Thm 6.11 (Amitsur) If R is a ~~semiprime~~ ring $\boxed{\text{that contains no nil ideals}}$ and T is a central indeterminate, then $R[T]$ is semiprimitive.

Proof: Suppose $J(R[T]) \neq 0$ and let $n \geq 1$ be the minimal degree of a nonzero element of $J(R[T])$. Define

$$I = \{ a_n \in R \mid \exists a_0, \dots, a_{n-1} \in R : a_n T^n + \dots + a_0 \in J(R[T]) \}.$$

$\Longrightarrow 0 \neq I \trianglelefteq R$.

If $f = \sum_{i=0}^{n} a_i T^i \in J(R[T])$, then $[f, a_n] \in J(R[T])$ and deg $[f, a_n] <$ deg f. Hence $[f, a_n] = 0$ and $\forall i \in [0,n]$: $[a_i, a_n] = 0$.

deg $[f,a_{n-1}] <$ deg f and $\quad J(R[T])$

~~But then $[f, a_{n-1}] \in J$ and so since $[f, a_{n-1}] = 0$. Hence $\forall i \in [0,n]$: $[a_i, a_{n-1}] = 0$.~~

~~Continuing inductively, we see that $a_0, \dots, a_n$ commute.~~

~~Then $1 - fT \in R[T]$~~

Since $1 - fT \in R[T]^\times$ by Thm 6.8, Prop. 6.10 implies that a_n is nilpotent.

$a_n \in I$ arbitrary

$\Longrightarrow I$ is nil. ↯

But then, for $i \in [0, n-1]$, deg $[f, a_i] <$ deg f and $[f, a_i] \in J(R[T])$, hence $[f, a_i] = 0$. Hence, $\forall i,j \in [0,n]$: $[a_i, a_j] = 0$. □

7. Posner's Theorem

Every commutative domain has a field of fractions, but not every noncommutative domain has a „division ring of fractions". Posner's Thm. provides the existence of such a quotient ring for prime PI rings. (Cf. also Goldie's Theorem)

~~Theorem 7.1~~

<u>Def 7.1</u> An ideal $P \trianglelefteq R$ is <u>primitive</u> if R/P is a primitive ring. ($\Leftrightarrow$ P is the annihilator of a simple right R-module).

<u>Theorem 7.2 (Rowen)</u> Let R be a (semi)prime PI ring and $0 \neq I \trianglelefteq R$. Then $I \cap Z(R) \neq 0$.

proved only for prime, since Thm 6.11 only for prime

<u>Proof:</u> $0 \neq I[T] \trianglelefteq R[T]$ and $Z(R[T]) = Z(R)[T]$.

It suffices to show $(I \cap Z(R))[T] = I[T] \cap Z(R)[T] \neq 0$.

Thus we may suppose that R is semiprimitive by Thm 6.11.

If $P \trianglelefteq R$ is primitive, then R/P is a CSA by Kaplansky's Theorem. Since R/P is simple, the image of I in R/P is either 0 or R/P. For at least one P, the image of I is R/P since $0 \neq I$ and $0 = \bigcap_{\substack{P \trianglelefteq R \\ \text{primitive}}} P$.

For each primitive $P \trianglelefteq R$, by Cor. 5.10, there exists a field L_P s.t. R satisfies the same multilinear identities as $M_{n_P}(L_P)$, with $n_P^2 = \dim_{Z(R)} R$, and in particular, g_{n_P} is a central polynomial for R/P.

Since R/P satisfies no monic PI of degree $< 2n_P$, and each R/P satisfies the same monic PI that R does, the set $\{n_P \mid P \trianglelefteq R \text{ primitive}\}$ is bounded.

If $m > n_P$, then $g_m(R/P) = 0$.

Thus there exists a maximal $m \in \mathbb{N}$ such that g_m is a central polynomial for some R/P in which the image of I is nonzero.

Then $g_m\left(\frac{I+P}{P}\right) \subset Z(R/P)$ for all primitive $P \triangleleft R$ (Lemma 4.11). (*)

Since $R \hookrightarrow \prod_{\substack{P \triangleleft R \\ P \text{ primitive}}} R/P$ ~~[illegible]~~, $g_m(I) \subset R \cap Z\left(\prod_P R/P\right) \subset Z(R)$,

and $g_m(I) \neq 0$.

((*): $m < n_P$: $\frac{I+P}{P} = 0$
$m = n_P$: ✓ g_m central
$m > n_P$: g_{n_P} [illegible] on R/P)

□

(*) !!!
Note that the center of a prime ring is a domain
[Let $a, b \in Z(R)$ with $ab = 0 \Rightarrow aRbR = 0 \Rightarrow aR = 0$ or $bR = 0 \Rightarrow a = 0$ or $b = 0$]

<u>Theorem 7.4 (Posner's Theorem)</u> Let R be a prime PI ring, $S := Z(R) \setminus \{0\}$, $K = S^{-1}Z(R)$ the field of fractions of $Z(R)$, and $Q = S^{-1}R$. Then Q is a CSA with center K, R is an order in Q, and $Q = KR$.

(*) <u>Def. 7.3</u> Let Q be a quotient ring (a ring in which every non-zero-divisor is invertible).
A subring $R \subset Q$ is an <u>order</u> if each $q \in Q$ can be written as $q = r_1 s_1^{-1} = s_2^{-1} r_2$ with $r_i, s_i \in R$.

<u>Proof</u>: Since $Z(R)$ is a commutative domain, the localization $Q = S^{-1}R$ exists and $R \hookrightarrow S^{-1}R$ (since R is a prime ring) and $K \hookrightarrow S^{-1}R$.
We can assume $R \subset S^{-1}R$ and $K \subset S^{-1}R$.
<u>Q is simple</u>: If $0 \neq I \triangleleft Q$, then $0 \neq I \cap R$. By Thm 7.2, $I \cap S \neq \emptyset$, hence $I = Q$. □ (Q simple)

As a central extension of the PI ring R, Q is a PI ring. By Kaplansky's Thm it is a CSA. $Z(Q) = K$ and $KR = Q$ is clear, and so is that R is an order in Q.

□

Cor 7.5 (1) A prime ~~PI~~ ring R is a PI ring if and only if it is an order in a CSA

[Actually, any order in a CSA is a prime PI ring]

(2) A prime PI ring is a bounded Goldie ring

(3) If R is a prime PI ring, R has the same PI class as its quotient ring.

Proof. (1)✓ (2) follows from (1) (without proof),

(3) Let $M_n(D)$ with $n\geq 1$ and D a division ring be the quotient ring of R. If K is the field of fraction of $Z(R)$, then $M_n(D)=RK$ is a central extension. Hence R and $M_n(D)$ satisfy the same multilinear identities. $\square$

8. Noetherian PI rings

Lemma 8.1 (1) There exists an n^2-alternating multilinear central polynomial for $M_n(A)$

$$\cancel{f\in\mathbb{Z}\langle X\rangle, m} \quad f(x_1,\dots,x_m)\in\mathbb{Z}\langle X\rangle,\ m\geq n^2.$$

(2) If f is as in (1), $\forall a_0,\dots,a_m\in M_n(A)$:

$$f(a_1,\dots,a_m)a_0=\sum_{i=1}^{n^2}(-1)^{i-1}f(a_0,\dots,a_{i-1},a_{i+1},\dots,a_m)a_i\in_{Z(R)}\langle a_1,\dots,a_m\rangle.$$

Proof: (1) $g_n(x_1,\dots,x_n)$ is a multilinear central polynomial for $M_n(A)$. Replacing x_1 by the Capelli polynomial $c_{n^2}(u_1,\dots,u_{n^2};v_1,\dots,v_{n^2})$, we obtain a multilinear n^2-alternating polynomial f. Clearly $f(M_n(A))\subset Z(M_n(A))$. Since $e_{ij}\in C_{n^2}(M_n(A))$ (cf. Prop. 4.6(3)) and g_n is x_1-linear, we also have $f(M_n(A))\neq 0$.

(2) $\tilde{f}(x_0,\dots,x_m):=\sum_{i=0}^{n^2}(-1)^i f(x_0,\dots,x_{i-1},x_{i+1},\dots,x_m)x_i$ is multilinear and n^2+1-alternating.

(

(Let $0 \le i < j \le n^2$ and substitute x_i for x_j. Then

$f(x_0, \dots, x_{i-1}, \underline{x_i}, x_{i+1}, \dots, x_{j-1}, \underline{x_i}, x_{j+1}, \dots, x_{n^2})$

$\underset{\substack{\uparrow \\ f\ n^2\text{-alternating}}}{=} (-1)^i f(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, \underline{x_i}, x_{j+1}, \dots, x_{n^2}) + (-1)^j f(x_0, \dots, x_{i-1}, \underline{x_i}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_{n^2})$

$= 0$ (since swapping x_i with $(j-1)-(i+1)+1 = j-i+1$ variables gives a sign $(-1)^{i-j+1}$).

□

Theorem 8.2 Let R be a prime PI ring of PI class t. Then there exists a $Z(R)$-submodule $F \subset R$, free of rank t^2, and a $c \in Z(R) \setminus \{0\}$ such that $cR \subset F$.

Proof: Let $M_n(D)$ be the quotient ring of R with D a division ring and $n \geq 1$. Let $L \subset D$ be a maximal subfield, so that

$$R \hookrightarrow M_n(D) \otimes_{Z(D)} L \cong M_t(L). \qquad \text{(cf. Cor 5.10)}$$

Without restriction $R \subset M_t(L)$ and then $RL = M_t(L)$.
Let $f(x_1, \dots, x_m)$ be a multilinear, t^2-alternating central polynomial for ~~$M_n(A)$~~ $M_t(L)$ (exists by Lemma 8.2(1)).
Let $a_1, \dots, a_{t^2}$ be an L-basis for $M_t(L)$. By multilinearity and non-vanishing of f on R, there exist $a_{t^2+1}, \dots, a_m \in R$ such that $c := f(a_1, \dots, a_m) \neq 0$.
By Lemma 8.2(2), $cR \subset \underbrace{\bigoplus_{i=1}^{m} Z(R) a_i}_{=: F}$.

□

Remark Using Thm 8.2, one can show that every prime PI ring R is even a $Z(R)$-order in its quotient ring.
(I.e., every $r \in R$ is <u>almost integral</u> over $Z(R)$).
This extends the characterization in Cor 7.5(1)

Theorem 8.3 (Cauchon) Let R be a prime PI ring.
TFAE:
(a) R satisfies the ascending chain condition (ACC) on ideals
(b) R is right Noetherian
(c) R is left Noetherian.

Proof: It suffices to show (a) $\Rightarrow$ (b).
Let $r_1, r_2, \ldots \in R$. We have to show $r_k \in \sum_{i=1}^{k-1} r_i R$ for $k \gg 1$.

By Thm 8.2, there exist $Z(R)$-linearly independent $a_1, \ldots, a_{t^2} \in R$ ~~and~~ $c \in Z(R) \setminus \{0\}$, and an injective map
$$\varphi : R \longrightarrow \bigoplus_{i=1}^{t^2} a_i Z(R) \cong Z(R)^{t^2}, \quad r \longmapsto cr = \sum_{j=1}^{t^2} \varphi_j(r) a_j$$
with suitable $\varphi_i : R \to Z(R)$.
Now $Z(R)^{t^2} \subset R^{t^2}$. ~~Since $_RR_R$ is~~ Since R satisfies the ACC on ideals, the (R,R)-bimodule R^{t^2} satisfies the ACC on subbimodules. Thus
$$\varphi(r_k) \in \sum_{i=1}^{k} R\varphi(r_i)R \underset{\uparrow}{=} \sum_{i=1}^{k-1} \varphi(r_i) R \quad \text{for } k \gg 1.$$
$\varphi(r_i) \in Z(R)^{t^2}$

Thus, $\varphi(r_k) = \sum_{i=1}^{k-1} \varphi(r_i) s_i$ for some $s_1, \ldots, s_{k-1} \in R$.
Hence, $\forall j \in [1, t^2]$: $\varphi_j(r_k) = \sum_{i=1}^{k-1} \overbrace{\varphi_j(r_i) s_i}^{= s_i \varphi_j(r_i)} \in R$
$$\Rightarrow \cancel{\varphi(r_k)} \; cr_k = \sum_{j=1}^{t^2} \varphi_j(r_k) a_j = \sum_{i=1}^{k-1} s_i \underbrace{\left(\sum_{j=1}^{t^2} \varphi_j(r_i) a_j \right)}_{= cr_i}$$
$$\Rightarrow r_k = \sum_{i=1}^{k-1} s_i r_i.$$
$\square$

Def. 8.4 Let M_A be a module. A submodule $N_A \subset M_A$ is *extended* if there exists $I \trianglelefteq A$ such that $N = IM$.

Lemma 8.5 Let $M = \langle m_1, \dots, m_k \rangle_A$ be a f.g. A-module.

(1) $\forall a \in A: \quad aM = 0 \iff \forall i \in [1,k]: \ am_i = 0$.

(2) If M is faithful, there exists a submodule $N \subset M$ maximal with respect to the property that M/N is faithful.

Proof: (1) ✓

(2) Let $\Omega := \{ N_A \subset M_A \mid M/N \text{ faithful} \}$.

Since $M = M/0$ is faithful, $\Omega \neq \emptyset$.

By Zorn's lemma it suffices to show:
If $N_1 \subset N_2 \subset \dots$ with M/N_i faithful, then $L := M/\bigcup_{j \geq 1} N_j$ is faithful.

Suppose $a \in A$ with $aL = 0$. Then $am_i \in \bigcup_{j \geq 1} N_j$ for all $i \in [1,k]$, hence there exists $j \geq 1$ s.t. $am_i \in N_j$ for all $i \in [1,k]$.

$(1) \Rightarrow aM/N_j = 0 \Rightarrow a = 0$.

$\square$

Thm 8.6 (Formanek)

(1) If A has a faithful Noetherian module M_A, then A is a Noetherian ring.

(2) If M_A is a faithful f.g. module satisfying the ACC on ~~ex~~ extended submodules, then M_A is Noetherian.
In particular: Then A is a Noetherian ring.

<u>Proof:</u> (1) Let $M = \langle m_1, \dots, m_k \rangle_A$. By Lemma 8.5(1),

$$A \hookrightarrow \prod_{i=1}^{k} A/\operatorname{ann}_A(m_i).$$

Since $A/\operatorname{ann}(m_i) \cong Am_i \subset M$, each $A/\operatorname{ann}_A(m_i)$ is Noetherian as A-module. Hence so is their finite product, and therefore also A.

(2) ~~If M_A is a faithful~~

Suppose M_A is not Noetherian. Let $IM \subset M$ with $I \trianglelefteq A$ be an extended submodule of M, maximal with respect to M/IM not being Noetherian. (ACC)

M/IM is a faithful $A/\operatorname{ann}_A(M/IM)$-module. ~~We may therefore~~ ~~Replacing M~~. We may therefore without restriction assume that for ~~on~~ every $0 \neq I \trianglelefteq A$, M/IM is Noetherian.

Let $N \subset M$ be maximal w.r.t. M/N being faithful (Lemma 8.5(2)). Let $N \subsetneq N' \subset M$. Then M/N' is not faithful. Let $a \in A \setminus \{0\}$ be such that $aM \subset N'$. Since M/aM is Noetherian, so is M/N'. Since $N' \supsetneq N$ was arbitrary, this implies that M/N is Noetherian. Since M/N is also faithful, (1) implies that A is Noetherian. Then M, as a f.g. A-module, is Noetherian.

The „in particular" follows from (1). $\square$

Cor 8.7 If R is a prime PI ring, TFAE
(a) $Z(R)$ is Noetherian
(b) $R_{Z(R)}$ is f.g. and R is right [left] Noetherian.

Proof: (a) $\Rightarrow$ (b): ~~$R_{Z(R)}$ is Z-Noetherian, and Z ... so is~~

By Thm 8.2, there is a $Z(R)$-monomorphism $R_{Z(R)} \hookrightarrow Z(R)^{t^2}$, $t \geq 1$.
Since $Z(R)^{t^2}$ is a Noetherian $Z(R)$-module, so is $R_{Z(R)}$.
Thus $R_{Z(R)}$ is f.g., and R_R is Noetherian.

(b) $\Rightarrow$ (a): $R_{Z(R)}$ is f.g. and faithful. Since R_R is right Noetherian it also satisfies the ACC on extended submodules (right ideals of the form IR_R for $I \triangleleft R$.) By Thm 8.6, $Z(R)$ is Noetherian. $\square$

Remark •) Prime PI rings also satisfy the DCC on prime ideals.
•) There exist Noetherian prime PI rings whose center is not Noetherian, or indeed, that are not even integral over the center.

Exm (McR5.3.7(iii)) A prime PI ring that is Noetherian but not integral over $Z(R)$ (Hence $Z(R)$ is not Noetherian)

Let L be a field, $\sigma, \tau \in Aut(L)$ of finite order and such that there exists $x \in L^\sigma$: $(\tau\sigma)^n(x)$ are all distinct for $n \in \mathbb{Z}$.

[E.g. K a field, char $K=0$, $L = K(x,y)$ $\sigma(x)=x$, $\sigma(y)=-y$, $\tau(x)=x+y$, $\tau(y)=-y$
Then ~~$(\sigma\tau)$~~ $(\tau\sigma)^n(x) = x+ny$]

$G := \langle \sigma, \tau \rangle \leq Aut(L)$

$$R := \begin{bmatrix} L^\sigma + TL[T] & TL[T] \\ TL[T] & L^\tau + TL[T] \end{bmatrix}$$

Then $Z(R) = (L^\sigma + TL[T]) \cap (L^\tau + TL[T]) = L^G + TL[T]$.

(i) R is an order in $M_2(L(T))$, hence a prime PI ring.

(ii) R is not integral over $Z(R)$:

Suppose $\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$ is integral over $Z(R)$. Then x is algebraic over L^G. $\Rightarrow$ $\{ (\cancel{\sigma\tau}\,\tau\sigma)^n(x) \mid n \in \mathbb{Z} \}$ are all roots of the same monic poly over L^G $\lightning$.

(iii) R is Noetherian:

Let $\pi \in \{\sigma, \tau\}$. Since $\dim_{L^\pi} L < \infty$, $L[T]$ is f.g. / $L^\pi[T]$.
Hence, so is $TL^\pi[T]$ and $L^\pi + TL[T]$.

~~In particular, $TL[T]$ is a Noetherian module over~~ [illegible]

Hence, $L^\pi + TL[T]$ is a Noetherian ring, and $TL[T]$ is a Noetherian module over it.

A basic result implies that R is Noetherian (McR 1.1.7)