Course Notes

Topics in algebra: Modules over hereditary noetherian prime rings

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Modules over hereditary noetherian prime (HNP) rings

References: [LR] Levy, Robson: Hereditary Noetherian Prime Rings and Idealizers, 2011

[L] Lam: Lectures on Modules and Rings, 1999

[GW] Goodearl, Warfield: An Introduction to Noncommutative Noetherian Rings, 2004

[MR] McConnell, Robson: Noncommutative Noetherian Rings, 2000

[AF] Anderson, Fuller: Rings and Categories of Modules, 1992

0. High-Level Overview/Introduction

Let R be a ring (associative, unital, possibly noncommutative).
A (right) R-module M_R is projective if it is a direct summand of a free module (i.e., $\exists N_R \in \text{Mod-}R$: $M \oplus N \cong R_R^{(I)}$ for a set I), if it is also finitely generated (f.g.), then I can be taken finite.

Def: (1) R is prime if $\forall I, J \trianglelefteq R$: $IJ = 0 \Rightarrow I = 0$ or $J = 0$
(2) R is right hereditary if every right ideal of R is projective (as right R-module), and hereditary if it is left & right hereditary.

Def: R is a hereditary noetherian prime (HNP) ring if it is (left and right) noetherian, hereditary and prime.

For commutative rings: prime $\Leftrightarrow$ domain, and commutative HNP rings turn out to be precisely the Dedekind domains.

Examples

(1) A commutative ring R is a HNP ring if and only if it is a Dedekind domain ($\Leftrightarrow$ every ideal factors uniquely into prime ideals $\Leftrightarrow$ every fractional ideal is invertible)

Examples: $\mathbb{Z}$, $K[x]$ with K a field, more generally: principal ideal domains (PIDs), $\mathbb{Z}[i]$

$\mathbb{Z}[\sqrt{-5}]$ ← not a PID, more generally: rings of integers in algebraic number fields, coordinate rings of nonsingular affine algebraic curves

[Actually: for commutative rings: R HNP $\Leftrightarrow$ R hereditary

(2) Let R be a principal right ideal domain (PRID).

By definition every $I \leqslant R_R$ is principal, i.e. $I = aR$ for $a \in R$.

Since R is a domain, if $a \neq 0$, the homomorphism

$$\begin{cases} R \longrightarrow aR \\ r \longmapsto ar \end{cases}$$

is an isomorphism, and aR is free (hence projective)

So every PRID is a right hereditary, right noetherian domain.

$\Rightarrow$ If R is a PRID and a PLID (principal left ideal domain), then R is a HNP ring.

(3) Let S be a ring and $\delta : S \to S$ a derivation, i.e.

$\forall a, b \in S$: $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = a\,\delta(b) + \delta(a)\,b$.

The skew polynomial ring $R = S[x; \delta]$ is the ring of all formal expressions of the form

$$\sum_{i=0}^{n} x^i s_i \qquad (n \geq 0,\ s_i \in S) \qquad \text{with}$$

coefficientwise addition and multiplication defined by $\boxed{sx := xs + \delta(s)}$

It is easy to check:

(i) if S is prime, so is R; if S is a domain, so is R

(ii) if S is left [right] noetherian, R is left [right] noetherian ("Hilbert basis theorem")

(iii) If S is a division ring, R is Euclidean (division with remainder using the degree function), and hence a PID.

[MR, (1.2.9)]

(a) Let $S = k(y)$ with k a field of characteristic 0,

$$B_1(k) := k(y)[x; \tfrac{d}{dy}]$$

Because $k(y)$ is a field, $B_1(k)$ is a PID, hence a HNP ring.

(b) $S = k[y]$, $\delta = \frac{d}{dy}$

$A_1(k) := k[y][x; \frac{d}{dy}] = k\langle x, y : yx - xy = 1\rangle$

is the first Weyl algebra.

It is not a PID, but a simple HNP ring (hereditary follows from [MR, (7.11.3), (7.11.8)] and is non-trivial), simplicity can be checked as an exercise

[Hint: first show $\forall f \in A_1(k)$: $[f, x] := fx - xf = \frac{\partial f}{\partial y}$

$[f, y] = fy - yf = -\frac{\partial f}{\partial x}$]

(c) Along these lines, if S is a commutative Dedekind domain, then $R = S[x; \sigma]$ is a hereditary noetherian domain

$\Leftrightarrow$ R is a simple ring. [MR, (7.11.2)]

The same result holds for skew Laurent polynomial rings $S[x^{\pm 1}; \sigma]$ with $\sigma \in \mathrm{Aut}(S)$.

(Here elements are of the form $\sum_{i=-m}^{n} x^i s_i$, $sx = x\sigma(s)$)

(4) Let D $\overset{\text{e.g. } D=\mathbb{Z}}{\swarrow}$ be a commutative Dedekind domain, $K=q(D)$ its quotient field.

Let A be a central simple K-algebra, i.e., a K-algebra that is (i) finite-dimensional as K-vector space,
(ii) simple as a ring,
(iii) has $Z(A)=K$.

(By <u>Artin-Wedderburn</u>, $A=M_n(E)$ for a division ring E with $\dim E_K<\infty$, $Z(E)=K$).

$$\begin{tikzcd} & A \arrow[d, dash] \\ R \arrow[ur, dash] \arrow[d, dash] & K \\ D \arrow[ur, dash] & \end{tikzcd}$$

A <u>(classical) D-order</u> is a subring $R\subseteq A$ such that
(i) $D\subseteq R$
(ii) R_D is finitely generated (as D-module)
(iii) $RK=A$ (i.e., R contains a K-basis of A)

A classical D-order is <u>maximal</u> if it is not properly contained in any other D-order.
Maximal D-orders are HNP rings! (non-trivial)

(i) $M_n(D)$ $(A=M_n(K))$ is a HNP ring.

(ii) Suppose <u>char $K\neq 2$</u>, let $a,b\in K^\times$.
The quaternion algebra $\left(\frac{a,b}{K}\right)$ is a 4-dimensional algebra with basis $1,i,j,k$ and relations

$$i^2=a,\quad j^2=b,\quad k^2=-ab \qquad ij=k=-ji$$

Every 4-dimensional CSA is isomorphic to a quaternion algebra, and 4 is the minimal dimension among noncommutative CSAs.

<u>E.g.</u> $A=\left(\frac{-1,-1}{\mathbb{Q}}\right)=\mathbb{H}$ is the $\mathbb{Q}$-algebra of Hamilton quaternions

$\mathcal{L}$ $\overset{\text{Lipschitz quaternions}}{\leftarrow}$ $={}_{\mathbb{Z}}\langle 1,i,j,k\rangle$ is a $\mathbb{Z}$-order in $\mathbb{H}$, but <u>not</u> maximal (and also not hereditary). The larger ring

$\mathcal{H} = \{a+bi+cj+dk : a,b,c,d \in \mathbb{Z} \text{ or } a,b,c,d \in \tfrac{1}{2}+\mathbb{Z}\}$

of <u>Hurwitz quaternions</u> is a maximal $\mathbb{Z}$-order (and a HNP ring)

<u>WARNING</u>: For a given algebra A, maximal orders are not unique (not even up to isomorphism)

This theory is developed in: Reiner: Maximal Orders, 1975
for quat. alg. see also: Voight: Quaternion Algebras, 2021.

Keep the notation of (4)

(5) There are hereditary D-orders that are <u>not</u> maximal, E.g., if $M \trianglelefteq D$ is maximal,

$$R = \begin{bmatrix} D & M \\ D & D \end{bmatrix}$$ is a hereditary D-order.

It arises as the <u>idealizer ring</u> of the maximal right ideal

$$\begin{bmatrix} M & M \\ D & D \end{bmatrix} \text{ in } M_2(D).$$

(6) In $S = A_1(k)$ (char $k=0$), xS is a maximal right ideal, the idealizer $\mathbb{I}_S(xS) = k + xS$ is also a HNP ring.

<u>Our goal is to understand something of the structure of R-modules, R HNP ring</u>

What do we know in commutative / very special cases?

We restrict to <u>finitely generated</u> modules.

(i) $R = K$ a field. K-modules $\triangleq$ K-vector spaces,
f.g. K-modules $\triangleq$ finite-dim. K-vector spaces

If M_K is a f.d. vector space, then $M = K_K^n$ with $n = \dim(M_K)$, and the dimension n characterizes M up to isomorphism

(ii) $R=\mathbb{Z}$ $\quad$ $\mathbb{Z}$-modules $\hat{=}$ abelian groups

Let M be a f.g. abelian group, $M_{tors} := \{x \in M : \exists n \in \mathbb{Z}\setminus\{0\}, nx=0\}$ the torsion subgroup. Then

$$M \cong M_{tors} \oplus M/M_{tors},$$

with M/M_{tors} f.g. torsion-free, hence $M/M_{tors} \cong \mathbb{Z}^n$ for some $\underset{\text{unique}}{n}$, and M_{tors} is a finite abelian group, with all indecomposable factors of the form $\mathbb{Z}/p^e\mathbb{Z}$ for a prime p, $e \geq 1$.

In particular: the $\mathbb{Z}/p^e\mathbb{Z}$ are uniserial (submodules form a chain), and all composition factors are isomorphic to each other (i.e. $\mathbb{Z}/p\mathbb{Z}$)

(iii) R is a commutative Dedekind domain (but not a field)

Let M_R be a f.g. R-module.

Again $M_R \cong M_{tors} \oplus M/M_{tors}$ with M/M_{tors} f.g. torsion-free (hence f.g. projective, but not in general free!)

- M_{tors} is a module of finite length, the indecomposable factors are of the form $R/\mathfrak{p}^e$ with $e \geq 1$, $0 \neq \mathfrak{p}$ a prime ideal of R.

  In particular, $R/\mathfrak{p}^e$ is uniserial and all composition factors are isomorphic (to $R/\mathfrak{p}$)

- $M/M_{tors} \cong I_1 \oplus \dots \oplus I_n$ with $n \geq 0$, $0 \neq I_j \trianglelefteq R$

  $$\cong R^{n-1} \oplus \underbrace{I_1 \cdots I_n}_{\trianglelefteq R} \quad (\text{if } n>0)$$

  here the rank n and the isomorphism class of the ideal $J := I_1 \cdots I_n$ characterize M/M_{tors} up to isomorphism.

Rem: The isomorphism classes of nonzero ideals form an abelian group, $\mathcal{C}(R)$ the class group of R

E.g. $\mathcal{C}(\mathbb{Z}) = 0$, but $\mathcal{C}(\mathbb{Z}[\sqrt{-5}]) = \mathbb{Z}/2\mathbb{Z}$

$$\{[(1)], [(2, 1+\sqrt{-5})]\}$$

So over $R = \mathbb{Z}[\sqrt{-5}]$, $R_R \oplus R_R$, $R_R \oplus \langle 2, 1+\sqrt{-5}\rangle_R$ are (up to isomorphism) the two torsion-free modules of rank 2.

(iv) There is a noncommutative notion of a Dedekind prime ring [MR, §5.2], these are the HNP rings that are also maximal orders in their quotient rings, e.g. $M_n(D)$, the, $A_1(k)$ but not $k + xA_1(k)$, $\begin{bmatrix} D & M \\ D & D \end{bmatrix}$.

Let R be a Dedekind prime ring, $0 \neq M_R$ a torsion-free module. Then $M_R \cong R^{n-1} \oplus I$ for some nonzero right ideal I of R.

The rank $(n-1)\,\mathrm{udim}\, R + \mathrm{udim}\, I$ is unique, but the isomorphism class is i.g. only unique up to stable isomorphism (I, J are stably isomorphic if $\exists m \geq 0$: $R^m \oplus I \cong R^m \oplus J$ $\overset{\text{Dedekind}}{\underset{\text{prime } R}{\iff}}$ $R \oplus I \cong R \oplus J$)

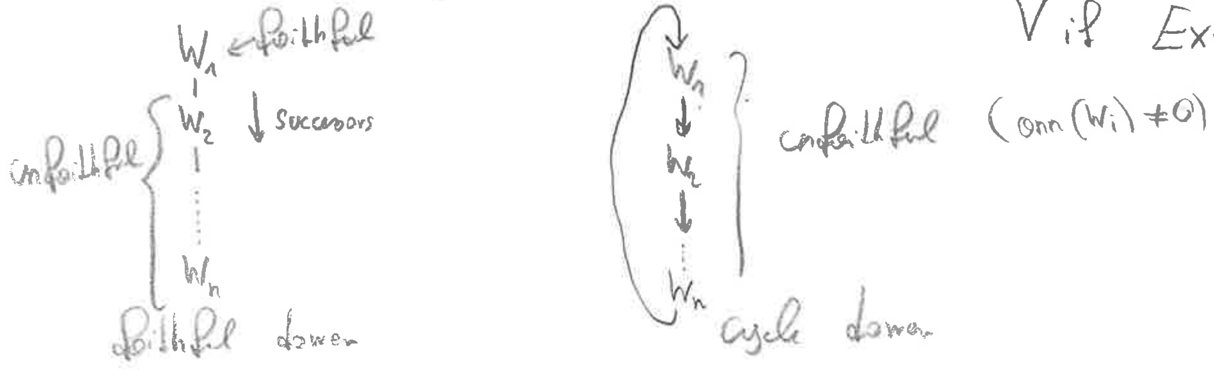
If $\mathrm{udim}\, M_R \geq 2$, the uniqueness is up to isomorphism. [MR, §5.7]

(v) HNP rings If M_R is f.g., then $M \cong M_{tors} \oplus M/M_{tors}$.

For torsion-free M, two invariants describe the stable isomorphism class of M (resp. the isomorphism class, if $\mathrm{udim}\, M \geq 2$):

- ideal class: element of an abelian group
- genus: a vector of non-neg. integers.

isoclasses of simple modules form towers: W is successor of V if $\mathrm{Ext}^1_R(V, W) \neq 0$



$\mathrm{modspec}(R) = \{\underline{0}\} \cup \{[V] : V_R \text{ simple, unfaithful}\}$

For P f.g. proj, one defines a rank of P at $V = 0$ or unfaithful simple, as $\rho(P,V) = \lambda\left(P/P\,\mathrm{ann}(V)\right)$ $(V \neq 0)$, $\rho(P,0) = \mathrm{udim}(P)$

↑ composition length, finite

The rank of P is $(\rho(P,V))_{[V] \in \mathrm{modspec}(R)}$. It satisfies two conditions

(i) for all but finitely many unfaithful $[V]$,

$$\rho(P,V) = \frac{\mathrm{udim}(P)}{\mathrm{udim}(R)}\,\rho(R,V) \qquad \text{(„almost standard rank")}$$

(ii) for every cycle tower $\mathcal{C}$:

$$\sum_{[V]\in\mathcal{C}} \rho(P,V) = \frac{\mathrm{udim}(P)}{\mathrm{udim}(R)} \sum_{[V]\in\mathcal{C}} \rho(R,V) \qquad \text{(„cycle standard rank")}$$

Subject to these conditions, genus and class are independent invariants describing M_P up to stable iso (up to iso, if udim $M_P \geq 2$)

Our goal is to establish (part of) this result!

1. Projective Modules, Hereditary Rings, Ext^1

We recall some properties of projective modules, partially w/o proofs. References are [L, §2], [AF, §17].

Let R be a ring.

Thm 1.1 For $P_R \in \text{Mod-}R$, TFAE:

(1) For every epimorphism $f: A_R \twoheadrightarrow B_R$ and every homomorphism $g: P_R \to B_R$, $\exists$ hom. $h: P_R \to A_R$ s.t. $g = f \circ h$.

$$\begin{tikzcd} & P \arrow[d, "g"] \arrow[dl, dashed, "h"'] & \\ A \arrow[r, "f"'] & B \arrow[r] & 0 \end{tikzcd}$$

(2) Every epi $f: A_R \twoheadrightarrow P_R$ splits, i.e., $\exists g: P_R \to A_R$ s.t. $f \circ g = id_P$

(3) Every SES $0 \to A \to B \to P \to 0$ is split exact

(4) P_R is a direct summand of a free module, i.e., $\exists Q_R: \; P_R \oplus Q_R = F_R$ with F_R free.

(5) $\mathrm{Hom}(P_R, -)$ is an exact functor $\text{Mod-}R \to \text{Mod-}R$, i.e., if $0 \to A_R \to B_R \to C_R \to 0$ is a S.E.S., then

$$0 \to \mathrm{Hom}(P_R, A_R) \to \mathrm{Hom}(P_R, B_R) \to \mathrm{Hom}(P_R, C_R) \to 0$$

is an exact sequence of abelian groups.

(Proof skipped.)

__Remarks__ 1) Suppose $P_R = \bigoplus_{i \in I} P_i$. Then P projective $\Leftrightarrow \forall i \in I$: P_i proj.

(Because the functor $\mathrm{Hom}(\oplus P_i, -)$ is equivalent to $\prod_{i \in I} \mathrm{Hom}(P_i, -)$, this follows from (5))

2) Free modules are projective

3) We write Proj-$R \subseteq$ Mod-R for the full subcategory of projective modules, proj-R for the full subcategory of f.g. projective modules

Dual Basis lemma

For $M_R \in$ Mod-R, let $M^* := \mathrm{Hom}(M_R, R_R)$ be the <u>dual</u> of M. M^* is naturally a <u>left</u> R-module, with

$$(rf)(m) := \overset{\text{makes sense in } {}_RR}{r f(m)} \qquad (r \in R,\ f \in M^*,\ m \in M),$$

<u>Lemma 1.2</u> (Dual Basis lemma) Let $P_R \in$ Mod-R

(1) P is projective <u>iff</u> there are families $(a_i)_{i\in I}$ in P, $(f_i)_{i \in I}$ in P^* s.t. for all $a \in P$ only finitely many $f_i(a) \neq 0$, <u>and</u>

$$a = \sum_{i \in I} a_i f_i(a) \qquad (*)$$

(2) P is f.g projective <u>iff</u> there are $a_1, \dots, a_n \in P$, $f_1, \dots, f_n \in P^*$ s.t.

$$\forall a \in P: \quad a = \sum_{i=1}^{n} \overbrace{a_i f_i}^{\in \mathrm{End}(M_R)}(a)$$

$$\left(\Leftrightarrow \quad id_P = \sum_{i=1}^{n} a_i f_i \quad \Leftrightarrow \quad id_P \in PP^* = \Big\{ \sum_{i=1}^{n} m_i g_i : n \geq 0,\ m_i \in M,\ g_i \in P^* \Big\} \subseteq \mathrm{End}(P_R) \right.$$

$$\left. \Leftrightarrow PP^* = \mathrm{End}(P_R) \right)$$

<u>Proof</u> (1) „$\Leftarrow$": Let F_R be free with basis $\{e_i : i \in I\}$.

$$f : \begin{cases} F \longrightarrow P \\ e_i \longmapsto a_i \end{cases} \text{ is an epimorphism,}$$

Define $g : P \longrightarrow F$ by $a \longmapsto \sum_{i \in I} e_i f_i(a)$. Because the f_i's are hom's, so is g. Since $f \circ g = id_P$ by $(*)$, f splits and $P \simeq g(P) \trianglelefteq F$.

„$\Rightarrow$" Fix a free F_R and an epi $f: F_R \longrightarrow P_R$.

By Thm 1.1(2), f splits. Let $g: P \to F$ s.t.

$f \circ g = id_P$. Expressing $F = \bigoplus_{i \in I} e_i R$, we can view $g = (e_i g_i)_{i \in I}$ with $g_i \in P^*$, i.e.,

$$\forall a \in P: \quad g(a) = \sum_{i \in I}' e_i g_i(a).$$

Then

$$a = f(g(a)) = f\Big(\sum_{i \in I}' e_i g_i(a)\Big) = \sum_{i \in I}' f(e_i g_i(a)) = \sum_{i \in I}' \underbrace{f(e_i)}_{= a_i} g_i(a).$$

(2) „$\Rightarrow$" If P_R is f.g. proj., we can take F in the previous construction to be f.g.

„$\Leftarrow$" $a_1, \dots, a_n$ generate P_R. $\square$

<u>Lemma 1.3</u> (1) If P_R is projective on n generators, then ${}_R P^*$ is projective on n generators.

(2) If P_R is projective, the canonical hom.

$$\Theta: \begin{cases} P_R \longrightarrow P^{**}_R \\ m \longmapsto \Theta_m \end{cases}, \quad \Theta_m(f) = f(m) \text{ is a monomorphism.}$$

If P_R is moreover f.g., then $P_R \cong P^{**}_R$ (P_R is <u>reflexive</u>)

<u>Proof:</u> (2) Check that Θ is a hom.

<u>Θ mono:</u> Let $m \in Ker(\Theta) \Rightarrow \forall f \in P^*: \Theta_m(f) = 0$

$\Rightarrow \forall f \in P^*: f(m) = 0$.

But $m = \sum_{i \in I}' a_i \underbrace{f_i(m)}_{=0}$ (Dual Basis Lemma), so $m = 0$.

<u>P_R f.g. $\Rightarrow \Theta$ epi:</u> Let $\varphi: P^* \longrightarrow R$.

$\forall g \in P^* \forall a \in P: \quad g(a) = \sum_{i=1}^n g(a_i) f_i(a)$ (Dual Basis Lemma),

so $\forall g \in P^*: \quad g = \sum_{i=1}^n g(a_i) f_i$.

$$\Rightarrow \varphi(g) = \varphi\Big(\sum_{i=1}^{n} \underbrace{g(a_i)}_{\in R} f_i\Big) = \sum_{i=1}^{n} g(a_i)\,\varphi(f_i).$$

Let $m := \sum_{i=1}^{n} a_i \varphi(f_i) \in M \Rightarrow \Theta_m(g) = g(m) = \varphi(g)$, so $\Theta(m) = \varphi$

(1) Consider the „dual basis" expression

$$\mathrm{id}_M = \sum_{i=1}^{n} \underset{\substack{\cap \\ M}}{a_i}\, \underset{\substack{\in \\ M^*}}{f_i}$$

Identifying $M \simeq M^{**}$ we get $\mathrm{id}_{M^{**}} \in M^{**} M^*$, and the symmetric version of Lemma 1.2 (2) shows ${}_R M^*$ is projective with generators $f_1, \dots, f_n$ $\square$

[Another way for (1): $M \trianglelefteq R_R^n$, so $M_R \oplus N_R \cong R_R^n$

$$\Rightarrow \underset{\substack{\| \\ (R_R)^* \cong {}_R R}}{\mathrm{Hom}(R_R, R_R)} \cong \underset{\substack{\| \\ (M_R)^* = {}_R M^*}}{\mathrm{Hom}(M_R, R_R)} \oplus \underset{\substack{\| \\ {}_R N^*}}{\mathrm{Hom}(N_R, R_R)} \Rightarrow {}_R R^n \cong {}_R M^* \oplus {}_R N^*]$$

Def: Let $M_R \in \mathrm{Mod}\text{-}R$. The trace ideal is

$$\mathrm{tr}(M_R) := M^* M := \Big\{ \sum_{i=1}^{n} \underbrace{f_i(a_i)}_{\in R} : f_i \in M^*, a_i \in M \Big\} = \sum_{f \in M^*} \mathrm{im}(f) \trianglelefteq R$$

Lemma 1.4 If P_R is projective, $T = \mathrm{tr}(P)$, then

$PT = P$, $T^2 = T$ (i.e. T is idempotent), $\mathrm{ann}(P) = \mathrm{rann}(T)$

Proof: Fix a „dual basis", so that

$$\forall a \in P: \quad a = \sum_{i \in I} \underset{\substack{\cap \\ P}}{a_i}\, \underset{\substack{\cap \\ T}}{\underline{f_i(a)}} \quad \Rightarrow PT = P \quad ✓$$

$T \subseteq T^2$: Let $f \in P^*$, $a \in P$, show $f(a) \in T^2$.

$$f(a) = \sum_{i \in I} \underbrace{f(a_i)}_{\in T} \underbrace{f_i(a)}_{\in T} \in T^2 \quad ✓$$

Annihilator: $PT = P \Rightarrow \mathrm{rann}(T) \subseteq \mathrm{ann}(P)$.

Let $r \in \mathrm{ann}(P) \Rightarrow \forall f \in P^* \forall a \in P$: $0 = f(\underbrace{ar}_{=0}) = f(a) r$

$\Rightarrow r \in \mathrm{rann}(T)$. $\square$

Hereditary rings

Def: R is <u>right hereditary</u> if every right ideal is projective

<u>Theorem 1.5</u> (Kaplansky) Let R be right hereditary.
Every submodule P of a free module F is a direct sum of right ideals.
In particular, P is projective.

Proof: Let $F = \bigoplus_{\alpha \in I} e_\alpha F$ and fix a well ordering on I (axiom of choice!).

For $\alpha \in I$, let $\begin{cases} F_{<\alpha} & \text{be the span of } e_\beta \text{ with } \beta < \alpha \\ F_{\leq\alpha} & \text{— " —————— } \beta \leq \alpha \end{cases}$

Every $a \in F_{\leq\alpha}$ has an expression $a = b + e_\alpha r$ with <u>unique</u> $b \in F_{<\alpha}$, $r \in R$.

$$\varphi: \begin{cases} P \cap F_{\leq\alpha} \longrightarrow R \\ a \longmapsto r \quad (\text{where } a = b + e_\alpha r) \end{cases} \text{ is a hom.,}$$

so $\varphi(P \cap F_{\leq\alpha}) =: J_\alpha \leq R_R$, $\ker(\varphi) = P \cap F_{<\alpha}$.

Because R is right hereditary, J_α is projective, and φ (with codomain J_α) splits, i.e.

$$P \cap F_{\leq\alpha} = (P \cap F_{<\alpha}) \oplus J_\alpha', \quad J_\alpha' \cong J_\alpha$$

<u>Claim</u>: $P = \bigoplus_{\alpha \in I} J_\alpha'$

<u>Proof of claim</u>: <u>directness</u>: Suppose $a_1 + \dots + a_n = 0$, $a_i \in J_{\alpha_i}'$,

$\alpha_1 < \dots < \alpha_n \implies a_1 + \dots + a_{n-1} \in F_{<\alpha_n}$, $a_n \in J_{\alpha_n}'$

$J_{\alpha_n}' \cap (P \cap F_{<\alpha_n}) = 0 \implies a_n = 0 \overset{\text{inductively}}{\implies} a_1 = \dots = a_{n-1} = 0$.

<u>$P = \sum_{\alpha \in I} J_\alpha'$</u>: Suppose $\sum_{\alpha \in I} J_\alpha' \subsetneq P$. Let $\beta \in I$ be minimal s.t.

$$P \cap F_{\leq\beta} \not\subseteq \sum_{\alpha \in I} J_\alpha'. \text{ Let } a \in (P \cap F_\beta) \setminus \sum_{\alpha \in I} J_\alpha'.$$

$\Rightarrow a = b + c, \quad b \in P \cap F_{<\beta}, \quad c \in J'_\beta$

$\Rightarrow b \in P \cap F_{\leq \gamma}$ for some $\gamma < \beta$

$\overset{\beta \text{ minimal}}{\Longrightarrow} \quad b \in \sum_{\alpha \in I} J'_\alpha, \quad c \in J'_\beta \Rightarrow a = b + c \in \sum_{\alpha \in I} J'_\alpha.$

Corollary 1.6 (1) Suppose R is right hereditary.

Then $P_R \in$ Mod-R is projective $\Leftrightarrow$ P_R embeds into a free module

(2) R right hereditary $\Leftrightarrow$

submodules of free modules are projective $\Leftrightarrow$

—— " —— projective —— " ——

(3) If R is a PRID, then submodules of free (right) modules are free.

Ext^n-functors: (brief reminder/crash course) [see e.g., Rodman, "An Introduction to Homological Algebra"; [LR, §5.3]]

Fix $M_R \in$ Mod-R.

$$\mathrm{Hom}(M_R, -): \underline{\text{Mod-}R} \to \underline{Ab}, \quad N_R \mapsto \mathrm{Hom}(M_R, N_R)$$

is a left exact functor, i.e., if

$$0 \to A_R \to B_R \to C_R \to 0$$

is exact, so is

$$0 \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C)$$

But $\mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C)$ is not surjective in general!

Long exact sequence:

$$0 \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \to$$
$$\mathrm{Ext}^1(M, A) \to \mathrm{Ext}^1(M, B) \to \mathrm{Ext}^1(M, C) \to$$
$$\mathrm{Ext}^2(M, A) \to \ldots$$

$\vdots$

$\mathrm{Ext}^i(M, -)$ are the right derived functors of $\mathrm{Hom}(M, -)$

How to compute them?

Proj. res. of M

$$\cdots \longrightarrow P_3 \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0 \qquad P_\bullet = (P_i, d_i)_{i\geq 0}$$

exact $\left(\operatorname{im}(d_i) = \ker(d_{i-1}),\ \operatorname{im}(\varepsilon) = M\right)$

Apply $\operatorname{Hom}(-, N)$ to $P_\bullet$

$$0 \longrightarrow \operatorname{Hom}(P_0, N) \xrightarrow{d_0^*} \operatorname{Hom}(P_1, N) \xrightarrow{d_1^*} \cdots$$ cochain complex $(d_i^* \circ d_{i-1}^* = 0)$

$$\operatorname{Ext}^i_R(M, N) := \ker(d_i^*) / \operatorname{im}(d_{i-1}^*)$$ i-th cohomology

Exm: p, q prime numbers, $\operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/q\mathbb{Z})$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_0(n) = np} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$
$$n \longmapsto n + p\mathbb{Z}$$
$$n \longmapsto np$$

$$0 \longrightarrow \underbrace{\operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/q\mathbb{Z})}_{\cong\ \mathbb{Z}/q\mathbb{Z}} \xrightarrow{d_0^*} \underbrace{\operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/q\mathbb{Z})}_{\cong\ \mathbb{Z}/q\mathbb{Z}} \longrightarrow \underbrace{\operatorname{Hom}(0, \mathbb{Z}/q\mathbb{Z})}_{=0} \to \underset{=0}{\cdots}$$

$$\varphi \longmapsto \varphi \circ d_0$$
$$(1 \mapsto a + q\mathbb{Z}) \longmapsto (1 \mapsto ap + q\mathbb{Z})$$

$$0 \longrightarrow \mathbb{Z}/q\mathbb{Z} \xrightarrow{d_0'} \mathbb{Z}/q\mathbb{Z} \longrightarrow 0$$
$$a + q\mathbb{Z} \longmapsto ap + q\mathbb{Z}$$

Case $p \neq q$: $\ker(d_0') = 0 \Rightarrow \operatorname{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/q\mathbb{Z}) = 0$,
$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/q\mathbb{Z}) = 0$

$p = q$: $d_0' = 0$, so
$\operatorname{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = \operatorname{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$

$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$

$\operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i \geq 2$

Remark: 1) M_R projective $\Leftrightarrow \forall N \in \text{Mod-}R: \; \operatorname{Ext}^1_R(M,N)=0$

$\Leftrightarrow \forall N \in \text{Mod-}R \, \forall i \geq 1: \; \operatorname{Ext}^i_R(M,N)=0$

[„$\Rightarrow$" $0 \to M_R \xrightarrow{id} M_R \to 0$ exact, „$\Leftarrow$" $\operatorname{Hom}(M_R,-)$ is exact]

2) R hereditary $\Rightarrow$ every M_R has proj. resolution of form

$$0 \to P_1 \to P_0 \to M \to 0$$

$\Rightarrow \operatorname{Ext}^i_R(M,N)=0$ if $i \geq 2$.

R has right global dimension ≤ 1

Ext^1 using extensions: Fix $M_R, N_R \in \text{Mod-}R$ [Rotman]

SES of the form $\mathcal{E}: 0 \to N_R \to X_R \to M_R \to 0$ are called extensions of N by M. $\mathcal{E}, \mathcal{E}'$ are equivalent if there exist a commutative diagram

$$\begin{tikzcd}
\mathcal{E}: & 0 \arrow[r] & N \arrow[r, "g"] \arrow[d, equal] & X \arrow[r, "h"] \arrow[d, "f", "\wr"'] & M \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{E}': & 0 \arrow[r] & N \arrow[r, "g'"] & X' \arrow[r, "h'"] & M \arrow[r] & 0
\end{tikzcd}$$

(Five Lemma: f hom. $\Rightarrow$ f iso.)

Equivalence classes form abelian group, isomorphic to $\operatorname{Ext}^1_R(M,N)$.

arbitrary extensions (not nec. equiv.)

$$\mathcal{E}+\mathcal{E}': \; 0 \to N \to \frac{\{(x,x') \in X \oplus X' : h(x)=h'(x')\}}{\{(-g(n), g'(n)) : n \in N\}} \to M \to 0$$ (Baer sum)

neutral element: $0 \to N \to N \oplus M \to M \to 0$ split extension.

Exm: $0 \to \mathbb{Z}/q\mathbb{Z} \to M \to \mathbb{Z}/p\mathbb{Z} \to 0$ p, q primes

$q \neq p$: Sequence splits, $M \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$

$\Rightarrow \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/q\mathbb{Z})=0$

<u>$p=q$:</u> $M \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ or $M \cong \mathbb{Z}/p^2\mathbb{Z}$

Every ext. $0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ splits ($\mathbb{Z}/p\mathbb{Z}$-vector space)

$$\mathcal{E}_{a,b}:\ 0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{g_a} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{h_b} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0 \quad \text{SES}$$

$$1+p\mathbb{Z} \longmapsto ap+p\mathbb{Z}$$
$$1+p^2\mathbb{Z} \longmapsto b+p\mathbb{Z}$$

a priori $p-1$ choices for a, b each ($a,b \not\equiv 0 \bmod p$)

Applying automorphism $1+p^2\mathbb{Z} \mapsto x+p^2\mathbb{Z}$ of $\mathbb{Z}/p^2\mathbb{Z}$, with $ax \equiv 1 \bmod p^2$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/p\mathbb{Z} \arrow[r, "g_a"] \arrow[d, equal] & \mathbb{Z}/p^2\mathbb{Z} \arrow[r, "h_b"] \arrow[d, "\cong"] & \mathbb{Z}/p\mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z}/p\mathbb{Z} \arrow[r, "g_1"] & \mathbb{Z}/p^2\mathbb{Z} \arrow[r] & \mathbb{Z}/p\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

$$1+p^2\mathbb{Z} \longmapsto x+p^2\mathbb{Z}$$

$$1+p\mathbb{Z} \longmapsto p+p\mathbb{Z}$$
$$1+p^2\mathbb{Z} \longmapsto ba+p\mathbb{Z}$$

so $\mathcal{E}_{a,b} \sim \mathcal{E}_{1,ab}$. There are exactly $p-1$ non-split extensions up to equivalence. [<u>Note</u>: Instead of „straightening" out g_a, we could do the same with h_b, but not both at the same time!]

<u>Module structure on $\mathrm{Ext}^n_R(M,N)$:</u> [see [LR, §53] for references]

Fix $n \geq 0$. $\mathrm{Ext}^n_R : (\underline{\mathrm{Mod}}\text{-}R)^{op} \times (\underline{\mathrm{Mod}}\text{-}R) \longrightarrow \underline{\mathrm{Ab}}$ are (bi)functors, contravariant in 1^{st} variable, covariant in 2^{nd} variable.

$\alpha_1 : M_1 \to M, \quad \varphi_1 : N \to N_1 \rightsquigarrow \mathrm{Ext}^n_R(M, \varphi_1) : \mathrm{Ext}^n_R(M,N) \to \mathrm{Ext}^n_R(M,N_1)$

$\varphi_1'' :=$

$\alpha_1' := \mathrm{Ext}^n_R(\alpha_1, N) : \mathrm{Ext}^n_R(M,N) \to \mathrm{Ext}^n_R(M_1,N)$

If $M_2 \xrightarrow{\alpha_2} M_1 \xrightarrow{\alpha_1} M$, $(\alpha_1 \circ \alpha_2)' = \alpha_2' \circ \alpha_1' : \mathrm{Ext}^n(M,N) \to \mathrm{Ext}^n(M_2,N)$

So, taking $M = M_1 = M_2$ ($\alpha_i \in \mathrm{End}(M_R)$) gives an $\mathrm{End}(M)^{op}$-left module structure on $\mathrm{Ext}^n(M,N)$.

Similarly, we get an $\mathrm{End}(N)^{op}$-right module structure on $\mathrm{Ext}^n(M,N)$

„Bifunctor commutativity" $\Longrightarrow$ $\mathrm{Ext}^n_R(M,N)$ is $(\mathrm{End}(M)^{op}, \mathrm{End}(N)^{op})$-bimodule.

Remark Hom's in the long exact sequence arising from $\mathrm{Hom}(M_R, -)$ are $\mathrm{End}(M)^{op}$-left modules, and analogously for $\mathrm{Hom}(-, N_R)$

In terms of extensions:

$$\mathcal{E}: \quad 0 \to N \to X \to M \to 0$$

Given $\alpha_1 : M_1 \to M$ define $\alpha_1 \mathcal{E}$:

$$\begin{tikzcd}
\alpha_1\mathcal{E}: 0 \arrow[r] & N \arrow[r, "\tilde{g}"] \arrow[d, equal] \arrow[dr, phantom, "//"] & X \times_M M_1 \arrow[r, "\tilde{h}"] \arrow[d, "f"] \arrow[dr, phantom, "//"] & M_1 \arrow[r] \arrow[d, "\alpha_1"] & 0 \\
\mathcal{E}: 0 \arrow[r] & N \arrow[r, "g"'] & X \arrow[r, "h"'] & M \arrow[r] & 0
\end{tikzcd}$$

pullback $X \times_M M_1$: $\{ (x, m_1) \in X \times M_1 : h(x) = \alpha_1(m_1) \}$

$\tilde{h}(x, m_1) = m_1$, $f(x, m_1) = x$

$\tilde{g}(n) = (g(n), 0)$ $\quad$ $\tilde{h}$ epi, $\tilde{g}$ mono ✓

(Well-definedness needs to be checked!)

⇝ $\mathrm{End}(M)^{op}$-left-module structure on $\mathrm{Ext}^1_R(M,N)$

$\mathrm{End}(N)^{op}$-right-module structure using pushouts.

Lemma 1.7 $\quad M_R, N_R \in \mathrm{Mod}\text{-}R$. If

(i) M_R is simple, $\mathrm{Ext}^1_R(M,N)$ has dimension 1 over $\mathrm{End}(M_R)^{op}$ (division ring) $\boxed{\text{OR}}$

(ii) N_R is simple, — " — $\mathrm{End}(N_R)^{op}$,

then, up to R-iso, there is a unique non-split extension module of N by M. (not a unique extension, just the middle module is unique!)

Proof: (i) ((ii) is similar) Let $\mathcal{E}, \mathcal{E}'$ be nonsplit extensions. (19)

$$\mathcal{E}: \quad 0 \to N \to X \to M \to 0$$

$$\mathcal{E}': \quad 0 \to N \to X' \to M \to 0$$

Show: $X_R \cong X'_R$.

$\dim_{\mathrm{End}(M_R)^{op}} \mathrm{Ext}^1_R(M,N) = 1 \Rightarrow \exists \alpha \in \mathrm{End}(M_R)$ s.t. $\mathcal{E} \sim \alpha\mathcal{E}'$, so w.r.t. $\mathcal{E} = \alpha\mathcal{E}'$,

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\gamma"] & M \arrow[r] \arrow[d, "\alpha", "\wr"'] & 0 & \alpha\mathcal{E}' = \mathcal{E} \\ 0 \arrow[r] & N \arrow[r] & X' \arrow[r] & M \arrow[r] & 0 & \mathcal{E}' \end{tikzcd}$$

γ hom, but Five lemma $\Rightarrow$ γ iso, so $X \cong X'$. □

Remark: Dimension shifting respects the module structures on Ext.

Let $0 \to A_R \to B_R \to C_R \to 0$ be a SES. $\forall n \geq 1$

(i) If B projective $\mathrm{Ext}^{n+1}_R(C,N) \underset{\uparrow}{\cong} \mathrm{Ext}^n_R(A,N)$ $\forall N \in \text{Mod-}R$

as $\mathrm{End}(N)^{op}$ modules

(ii) — " — injective, $\mathrm{Ext}^{n+1}_R(M,A) \underset{\uparrow}{\cong} \mathrm{Ext}^n(M,C)$ $\forall M \in \text{Mod-}R$,

as $\mathrm{End}(M)^{op}$-modules

(for (i). $0 \to \mathrm{Hom}(C,N) \to \mathrm{Hom}(B,N) \to \mathrm{Hom}(A,N) \to$

$\to \mathrm{Ext}^1(C,N) \to \mathrm{Ext}^1(B,N) \to \mathrm{Ext}^1(A,N) \to$

$\to \mathrm{Ext}^2(C,N) \to 0 \to \mathrm{Ext}^2(A,N) \to \dots$)

2. Quotient Rings [L, Ch. 9], Uniform Dimension

2.1 Ore condition & rings of quotients

Given a commutative ring R and a multiplicative set $S \subseteq R$ ($1 \in S$, $SS \subseteq S \not\ni 0$), we can form the quotient ring (or ring of fractions)

$RS^{-1} = \{\frac{r}{s} : r \in R, s \in R\}$ where $\frac{r}{s}$ are equivalence classes on $R \times S$:

$$(r, s) \sim (r', s') \iff \exists t \in S: \quad rs't = r's t$$

and operations $\frac{r}{s} \cdot \frac{r'}{s'} := \frac{rr'}{ss'}$, $\frac{r}{s} + \frac{r'}{s'} := \frac{rs' + r's}{ss'}$.

$j : R \to RS^{-1}$, $r \mapsto \frac{r}{1}$ is a ring hom with $\ker(j) = \{r \in R : \exists s \in S, rs = 0\}$

Special cases: • $S = R^{\circ} = \{r \in R : r \text{ not a zero-divisor}\}$

$Q(R) := RS^{-1}$ is the total ring of quotients $R \hookrightarrow Q(R)$

• $S = R^{\circ} = R \setminus \{0\}$: $Q(R)$ is the quotient field of R

Universal Property: For any ring hom $\varphi : R \to T$ s.t. $\varphi(S) \subseteq T^{\times}$, there is a unique ring hom $\overline{\varphi} : RS^{-1} \to T$ s.t. $\varphi = \overline{\varphi} \circ j$.

For noncommutative rings several things can fail:

*) Not every domain embeds in a division ring, e.g. if K is a field

$$K\langle a, b, c, d, x, y, u, v : ax = by, cx = dy, au = bv\rangle \text{ does not}$$

(Mal'cev) [L, Theorem 9.8]

*) There is always a ring satisfying the UP, but $\ker(j)$ may be big, (e.g., $R = M_n(K)$, $S = \{1, e_{11}\} \to RS^{-1} = 0$),

This ring may not be a div. ring even when R is a domain and/or elements may not be representable as fractions.

Def: R ring, $S \subseteq R$ multiplicative set.
R' is a <u>right quotient ring</u> (with respect to $S \subseteq R$) if there is a hom $j: R \to R'$ s.t.

(i) $j(S) \subseteq (R')^{\times}$

(ii) $R' = \{ j(r) j(s)^{-1} : r \in R, s \in S \}$

(iii) $\ker(j) = \{ r \in R : \exists s \in S : rs = 0 \}$

<u>**Theorem 2.1**</u> R, S as above.
A right quotient ring exists IFF

(i) $\forall a \in R \, \forall a \in S: \quad aS \cap bR \neq \emptyset$ (<u>right Ore condition</u>), and

(ii) $\forall a \in R: \left(\exists s' \in S: \quad s'a = 0 \right) \Rightarrow \exists s \in S: as = 0$ (<u>right reversible</u>)

S satisfying (i) & (ii) is called <u>right denominator set</u>.

(w/o proof)

<u>Remarks</u>: 1) Right quotient rings are unique up to unique isomorphism (use UP), we write RS^{-1} or $\varrho(R)$ if $S = R^{\circ}$.

2) The usual construction using equivalence classes works;

$$(r_1 s_1^{-1})(r_2 s_2^{-1}) := ?$$

$$r_2 S \cap s_1 R = \emptyset \Rightarrow r_2 \tilde{s} = s_1 \tilde{r} \quad (\tilde{s} \in S, \tilde{r} \in R), \quad s_1^{-1} r_2 = \tilde{r} \tilde{s}^{-1}$$

$$\boxed{(r_1 s_1^{-1})(r_2 s_2^{-1}) := (r \tilde{r})(s_2 \tilde{s})^{-1}}$$

$$r_1 s_1^{-1} + r_2 s_2^{-1} := ?$$

$$s_1 R \cap s_2 S \neq \emptyset \Rightarrow \underbrace{s_1}_{\in S} s = s_2 \underbrace{t}_{\in S} \in S, \quad \begin{aligned} r_1 s_1^{-1} &= r_1 s (s_1 s)^{-1} \\ r_2 s_2^{-1} &= r_2 t (s_2 t)^{-1} \end{aligned}$$

$$\boxed{r_1 s_1^{-1} + r_2 s_2^{-1} := (r_1 s + r_2 t)(s_1 s)^{-1}}$$

(Plenty of things to check!)

(*) 6) any finite set $q_1, \dots, q_n \in Q$ has a common right denominator. (as in 2)

3) If right & left quotient rings exist, they are isomorphic (UP)

4) R is a <u>right Ore domain</u> if R is a domain & R° satisfies right Ore condition

5) E.g. $M_n(\mathbb{Z})$, $S = \mathbb{Z}^{\circ} \rightsquigarrow M_n(\mathbb{Q})$, also $S = M_n(\mathbb{Z})^{\circ} \rightsquigarrow M_n(\mathbb{Q})$; $\varrho\left(\begin{bmatrix} \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}\right) \cong M_2(\mathbb{Q})$ (*)

2.2 Uniform Dimension

Let R be a ring,

Def: For $M_R \in \text{Mod-}R$, the uniform dimension (also: Goldie dimension) $\text{udim}\, M_R \in \mathbb{N}_0 \cup \{\infty\}$ is the supremum over all $n \in \mathbb{N}_0$ s.t. M contains a direct sum $M_1 \oplus \dots \oplus M_n$, $0 \neq M_i \leq M_R$.

Examples: •) $R = K$ field, $\text{udim}\, M_K = \dim V_K$ (vector space dimension)

•) $\text{udim}(\mathbb{Z}^n_{\mathbb{Z}}) = n$

•) $\text{udim}(\mathbb{Z}/p^n\mathbb{Z}) = 1$, $n = p_1^{e_1} \cdots p_r^{e_r}$, p_i distinct primes, $e_i \geq 1$

$\Rightarrow \text{udim}(\mathbb{Z}/n\mathbb{Z}) = r$

•) $\text{udim}\, M_n(\mathbb{Q})_{\mathbb{Q}} = n^2$, $\text{udim}\, M_n(\mathbb{Q})_{M_n(\mathbb{Q})} = n$ $\left[M_n(\mathbb{Q}) = \begin{bmatrix} \mathbb{Q} & \cdots & \mathbb{Q} \\ & 0 & \end{bmatrix} \oplus \begin{bmatrix} 0 & \cdots & 0 \\ \mathbb{Q} & \cdots & \mathbb{Q} \\ & 0 & \end{bmatrix} \oplus \dots \right]$

•) If R is a domain: R right Ore $\Leftrightarrow \text{udim}\, R_R = 1$

["$\Rightarrow$" $0 \neq I, J \leq R$. Show: $I \cap J \neq 0$. Let $a \in I \setminus \{0\}$, $b \in J \setminus \{0\}$

$\Rightarrow aR \cap \underbrace{bR^\bullet}_{\subseteq R^\bullet} \neq \emptyset \Rightarrow aR \cap bR \neq 0$.

"$\Leftarrow$" Let $a \in R^\bullet$, $b \in R$. Show: $aR \cap bR^\bullet \neq \emptyset$

Case 1: $b = 0$: $a \cdot 0 = b \cdot 1 = 0$ ✓

Case 2: $b \neq 0$: $\text{udim}\, R_R = 1 \Rightarrow aR \cap bR \neq 0 \Rightarrow \exists x, y \in R: ax = by \neq 0$ $(\Rightarrow y \neq 0)$]

•) $\text{udim}\, M_R = 0 \Leftrightarrow M = 0$

•) In $R = \mathbb{Q}\langle x, y \rangle$: $\bigoplus_{i \geq 0} y^i x R$ is direct $\Rightarrow \text{udim}\, R_R = \infty$.

($\Rightarrow \mathbb{Q}\langle x, y \rangle$ is a domain that is not right Ore)

Prop 2.2 Let R be a domain. Then either R is right Ore or it contains an infinite direct sum of nonzero right ideals.

In particular: Right noetherian domains are right Ore

Proof: Suppose R is not right Ore. Let $a, b \in R^\bullet$ s.t. $aR \cap bR = 0$. We claim $\sum_{i \geq 0} a^i bR$ is direct:

$\sum_{i=0}^n a^i b r_i = 0$, $r_i \in R$, wlog $r_0 \neq 0$, $\Rightarrow b r_0 = \underbrace{-\sum_{i=1}^n a^i b R}_{\neq 0} \in aR$.

Exm: $A_1(k)$ (even $A_n(k)$) is an Ore domain

Def: Let $M_R \in$ Mod-R

(1) $N_R \leq M_R$ is <u>essential</u> (notation: $N \leq_e M$) if

$$\forall 0 \neq K_R \leq M_R: \quad N_R \cap K_R \neq 0$$

(2) $0 \neq M_R$ is <u>uniform</u> if every nonzero submodule is essential

$\Leftrightarrow$ udim $M_R = 1$

$\Leftrightarrow \forall 0 \neq N_R, K_R \leq M_R: \quad N_R \cap K_R \neq 0$

<u>Properties</u>: [L, §6A,B, §3D], [MR, §2.2]

•) udim $M_R = n < \infty \Leftrightarrow \exists$ uniform $U_1, \dots, U_n \leq M$: $U_1 \oplus \dots \oplus U_n \leq_e M$,

udim $M_R = \infty$ IFF no such n exists.

•) udim $M = \infty \Leftrightarrow M$ contains an <u>infinite</u> direct sum of submodules

["$\Rightarrow$" is non-trivial: by contradiction. One first shows every $0 \neq N \leq M_R$ contains a uniform submodule]

•) M_R noetherian or artinian $\Rightarrow$ udim $M_R < \infty$

•) udim $\left(\bigoplus_{i=1}^{k} M_i\right) = \sum_{i=1}^{k}$ udim (M_i)

•) $N \leq M \Rightarrow$ udim $N \leq$ udim M, and

i) $N \leq_e M \Rightarrow$ udim $N =$ udim M ["essential" is transitive]

ii) Unless udim $N =$ udim $M = \infty$,

udim $N =$ udim $M \Rightarrow N \leq_e M$

•) ⚠ udim $M/N \not\leq$ udim M, e.g. udim $\mathbb{Z}_\mathbb{Z} = 1$, but udim $\mathbb{Z}/p_1 \cdots p_r\mathbb{Z} = r$, when $p_1, \dots, p_r$ are distinct primes, udim $\mathbb{Q}_\mathbb{Z} = 1$, udim $(\mathbb{Q}/\mathbb{Z})_\mathbb{Z} = \infty$

$\Rightarrow$ udim is <u>not</u> additive on SES!

•) <u>Localization</u>: Suppose $S = R^\circ$ is a right Ore set, $M_R \in$ Mod-R

We can construct the localization/quotient module MS^{-1} analogous to RS^{-1} (equivalence relation on $M \times S$), the natural map $j: M \to MS^{-1}$ is universal w.r.t. to R-hom's from M_R to RS^{-1}-modules.

$\ker(j) = \{m \in M : \exists s \in S: ms = 0\}$ is the torsion submodule (w.r.t. S), M is <u>torsion-free</u> if $\ker(j) = 0$. [$\forall m \in M\ \forall r \in R^\circ$: $mr = 0 \Rightarrow m = 0$]

$MS^{-1} \cong M \otimes_R RS^{-1}$ (naturally).

If M_R is $\boxed{\text{torsion-free}}$, then

$$\mathrm{udim}_R(M_R) = \mathrm{udim}_{RS^{-1}}(MS^{-1})$$

(E.g. $\mathrm{udim}\, \mathbb{Z}^2_{\mathbb{Z}} = 2 = \mathrm{udim}\,(\underbrace{\mathbb{Z}^2_{\mathbb{Z}} \otimes \mathbb{Q}}_{=\mathbb{Q}^2})$, but $(\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}) \otimes_{\mathbb{Q}} = \mathbb{Q}$!)

So udim behaves well w.r.t. localization:

__Lemma 2.3__ Let R be a right Ore ring (i.e, R° right Ore set), $Q = q(R)$

Let $I \leqslant R_R$, $J \leqslant Q_Q$. Then:

(i) $I \leqslant_e R_R \iff IQ \leqslant_e Q_Q$, $J \leqslant_e Q_Q \iff J \cap R \leqslant_e R_R$

(ii) $\mathrm{udim}\, I_R = \mathrm{udim}\, IQ_Q = \mathrm{udim}\, IQ_R$

(iii) $\mathrm{udim}\, J_Q = \mathrm{udim}\, J_R = \mathrm{udim}\,(J \cap R)_R$

In particular: $\mathrm{udim}\, R_R = \mathrm{udim}\, Q_Q$

__Def:__ 1) R has __ACC on right annihilators__ if every ascending chain of right ideals of the form $\{\mathrm{rann}(X) : \emptyset \neq X \subseteq R\}$ stabilizes

2) R is __right Goldie__ if $\mathrm{udim}\, R_R < \infty$ and it satisfies ACC on right annihilators

Right noetherian rings are (obviously) right Goldie!

__Theorem 2.4__ (Goldie's Theorem) [L, §11B] A ring R has a __semisimple ring of__ right quotients $Q = q(R)$ __iff__ R is semiprime right Goldie

Then: Q simple artinian $\iff$ R prime , Q division ring $\iff$ R domain

__Moreover:__ Q_R is a max. essential extension (so Q_R is an injective hull of R_R)

__Lemma 2.5__ (Goldie's Lemma) Let R be semiprime right Goldie, $I \leqslant R_R$.

__Then:__ $I \leqslant_e R_R \iff I \cap R^\circ \neq \emptyset$

__We shall need:__ R noetherian prime ring

$\implies Q$ simple artinian.

Examples: 1) $R = M_n(\mathbb{Z}) \subseteq M_n(\mathbb{Q}) = Q$, $\quad$ codim $R_R = n =$ udim Q_Q

$$R^\circ = \{A \in M_n(\mathbb{Z}) : \det(A) \neq 0\}$$

$$R(R^\circ)^{-1} = R(\mathbb{Z}^\circ)^{-1} = (R^\circ)^{-1}R = (\mathbb{Z}^\circ)^{-1}R$$

$$\begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ 3\mathbb{Z} & 3\mathbb{Z} \end{bmatrix} \leq_e M_2(\mathbb{Z}), \quad \left\{\begin{bmatrix} x & y & z \\ x & y & z \\ x' & y' & z' \end{bmatrix} : x, y, z, x', y', z' \in \mathbb{Z}\right\} \leq M_3(\mathbb{Z})$$

is not essential.

2) $A_1(K)$ (even $A_n(K)$) has a quotient div. ring, but, if char $K = 0$, $Z(A_n(K)) = K$, so it's not enough to invert central elements.

3) $q(Q) = Q$ if $\underline{Q \text{ is simple artinian}}$:

Then $Q \cong M_n(D)$, $n \geq 1$, D division ring, $n =$ udim Q_Q.

(Artin-Wedderburn)

.) Q has, up to iso, a unique simple (right) module $S = [D\ D \dots D] = D^{1\times n}$

.) Every $M \in$ Mod-Q is a direct sum of copies of S, with cardinality of the index set uniquely determined.

.) Every $M \in$ Mod-Q is projective & injective.

.) If $M \in$ Mod-Q, $N \leq M_Q \Rightarrow \exists K \leq M_Q : \quad M_Q = N \oplus K$

.) $Q^\circ \subseteq Q^\times$

(*)

Def: Let R be a right Ore ring, Q its right quotient ring.

(1) A $\underline{\text{fractional right } R\text{-ideal}}$ is a $I_R \leq Q_R$ s.t.

$\exists a, b \in R^\circ : \quad a \in I \ \wedge \ bI \subseteq R_R$

(2) A $\underline{\text{(integral) right } R\text{-ideal}}$ is a frac. right R-ideal I with $I \subseteq R$.

(3) If I, J are fractional right R-ideals, let

$$(I :_e J) := \{q \in Q : qJ \subseteq I\}, \quad O_e(I) := (I :_e I), \quad I^{-1} := (R :_e I)$$

left order

Remark

(1) Let R be semiprime [prime] right Goldie. A subring $T \subseteq Q = Q(R)$ ✻ 25b

is equivalent to R, if $\exists a, b, c, d \in Q^{\times}$: $aRb \subseteq T$ & $cTd \subseteq R$.

Any such ring is also semiprime [prime] right Goldie by the „easy" direction of Goldie's Theorem. Notation: $R \sim S$

(2) If S is a ring s.t. $R \subseteq S \subseteq Q$, S is semiprime right Goldie (and prime if R is prime)

If R semiprime right Goldie: right R-ideals $\hat{=}$ essential right ideals

R semiprime noetherian: frac. right R-ideals $\hat{=}$ f.g. R-submodules I of Q_R with $\mathrm{udim}(I) = \mathrm{udim}(Q)$

<u>Lemma 2.5(b)</u> Let R be a right Ore ring, $Q = q(R)$, $I, J \subseteq Q$ frac. right R-ideals

(1) $\mathrm{Hom}_R(I, J) \hookrightarrow \mathrm{Hom}_R(IQ, JQ)$

(2) $\mathrm{Hom}_R(I, J) \overset{\wedge}{\cong} (J :_e I)$ as left $O_e(J)$-modules, $I^* \cong I^{-1} := (R :_e I)$ as left R-modules, $\mathrm{End}(I_R) \cong O_e(I)$ as rings

(3) I_R is f.g. projective $\Leftrightarrow$ $I I^{-1} = O_e(I)$ („internal dual basis lemma")

<u>Proof</u>: (1) $\varphi \in \mathrm{Hom}(I, J)$ extends to $\tilde{\varphi}(rs^{-1}) = \varphi(r) s^{-1}$.
because $I \subseteq IQ$, the extension map is injective

(2) $(J :_e I) = \{ q \in Q : qI \subseteq J \} \longrightarrow \mathrm{Hom}_R(I, J)$
$q \longmapsto (x \in I \mapsto qx \in J)$

Only surjectivity is non-trivial. Let $\varphi \in \mathrm{Hom}_R(I, J)$, fix $a \in I \cap R^\circ$, extend φ to $\varphi : IQ \to JQ$. Then

$$\varphi(a) = \underbrace{\varphi(a) a^{-1}}_{=: b \in Q} a$$

<u>Claim</u>: $\forall x \in I: \ \varphi(x) = bx \ \wedge \ b \in (J :_e I)$

$\varphi(x) = \varphi(\underset{\in I}{a a^{-1} x}) = \varphi(a) a^{-1} x = b a a^{-1} x = bx$ ✓

$\varphi(I) \subseteq J \Rightarrow b \in (J :_e I)$

(3) We know I_R f.g. proj. $\Leftrightarrow$ $I I^* = \mathrm{End}(I_R)$ (Lemma 1.2(2))
After identifications $\Leftrightarrow$ $I I^{-1} = O_e(I)$

<u>Lemma 2.6</u> Let R be a <u>prime</u> right Goldie ring, $Q = q(R)$

(1) $I_R \leq R_R$ uniform $\Leftrightarrow$ $IQ \leq Q_Q$ minimal (nonzero) right ideal.

(2) If $U_R, V_R \leq R_R$ are uniform, then U_R is isomorphic to a submodule of V_R

(3) R_R has minimal right ideals $\Leftrightarrow$ $R = Q$

Proof: (1) $\mathrm{udim}\, IQ_Q = \mathrm{udim}\, I_R$ shows I_R uniform $\Leftrightarrow$ $\mathrm{udim}\, IQ = 1$.

Suffices to show: $\mathrm{udim}(IQ) = 1 \Rightarrow IQ$ minimal. („$\Leftarrow$" is trivial)

Suppose $0 \subsetneq J_Q \subsetneq IQ \xrightarrow{Q \text{ semisimple}} IQ \cong J_Q \oplus K_Q$, $0 \neq K_Q$ ↯.

(2) Because R is prime, $VU \neq 0$. Let $v \in V$ with $vU \neq 0$

$\Rightarrow \varphi: U \to V$, $x \mapsto vx$ is a nonzero hom.

$\overset{2.5(1)}{\Longrightarrow} \tilde{\varphi}: UQ \to VQ$ is nonzero $\xrightarrow{UQ \text{ simple}} \tilde{\varphi}$ mono $\Rightarrow \varphi$ mono.

(3) „$\Leftarrow$" ✓, „$\Rightarrow$" $d := \mathrm{udim}\, R_R$, Suppose $\exists\, \forall\, 0 \neq I$ minimal right ideal ($\Rightarrow I$ simple)

R contains an essential right ideal $E \leq_e R_R$ s.t.

$E = I_1 \oplus \dots \oplus I_d$, $I_j \leq R_R$ uniform (Properties of udim, p.23)

Each I_j is subisomorphic to I (by (2)), so $I_j \cong I$ $\forall j$. by simplicity of I.

$\Longrightarrow E \cong \underbrace{I \oplus \dots \oplus I}_{d \text{ times}} \Rightarrow Q_e(E) \overset{2.5(2)}{\cong} \mathrm{End}(E_R) \cong M_d(D)$, $D = \mathrm{End}(I_R)$ div. ring

So $Q_e(E)$ is simple artinian. Let $a \in E \cap R^\circ$ (Goldie's Lemma)

$Q_e(E)a \subseteq E \subseteq R$, $aR \subseteq E \subseteq Q_e(E)$ ($\Rightarrow R \sim Q_e(E)$)

Then $Q = Q_e(E) \Rightarrow Qa \subseteq R \Rightarrow R = Q$ is simple artinian $\square$

$\boxed{\text{We usually assume } R \neq q(R) \text{ to avoid trivial cases}}$

3. Basic module theory of HNP rings

Let R be an HNP ring, $Q = q(R)$

Lemma 3.1 If $0 \neq M_R$ is f.g. torsion-free, then there exist $P_R \leq M_R$ s.t. $M/P \cong U_R$ with $U_R \leq R_R$ uniform

Proof: $M_R \hookrightarrow M_R \otimes_R Q = MQ \subseteq Q^n$, clearing denominators on the left wlog $M \subseteq R_R^n$.

Recall Q_Q is semisimple

$\lambda(MQ) = \mathrm{udim}(MQ_Q) = \mathrm{udim}(M_R) =: d$

Let $0 \neq N_Q \subseteq MQ_Q$ be simple $\Rightarrow MQ_Q \cong N_Q \oplus K_Q$, $\lambda(K_Q) = d-1$

Let $\pi: MQ \twoheadrightarrow N$, $\ker(\pi) = K$

$\Rightarrow \ker(\pi|_M) = M \cap K \subsetneq M$ (bec. $\mathrm{udim}(M \cap K) < \mathrm{udim}(M)$)

$\Rightarrow 0 \neq \operatorname{im}(\pi|_M) \subseteq N_Q \Rightarrow U := \operatorname{im}(\pi|_M)$ is uniform submodule of N_R

Now $U_R \subseteq N_R \hookrightarrow Q_R$. U_R f.g. $\Rightarrow \exists d \in R^\cdot : U \cong dU \subseteq R_R$.

$\square$

<u>Prop 3.2</u> Every f.g. M_R is a direct sum of a torsion-free module and a torsion module: $M_R \cong \operatorname{tor}(M) \oplus M/\operatorname{tor}(M)$

<u>Proof:</u> $M' := M/\operatorname{tor}(M)$ is torsion-free & $\operatorname{udim}(M') \overset{R_R \text{ noeth}}{<} \infty$

$\Rightarrow$ wlog: $M' \subseteq R_R^n$ for some $n \geq 0$ $[\ M' \subseteq M' \otimes_R Q \subseteq Q^n,\ M'$ f.g. $\Rightarrow$ $\exists d \in R^\cdot: \ dM' \subseteq R^n\ ]$

$\overset{R \text{ right hered}}{\Longrightarrow} M'$ projective

$\Rightarrow 0 \to \operatorname{tor}(M) \to M \to M' \to 0$ splits. $\square$

<u>Thm 3.3</u> For a f.g. M_R. TFAE:

(a) M_R is torsion-free
(b) M_R is projective
(c) M_R is isomorphic to a direct sum of uniform right ideals of R

<u>In particular</u>: Right ideals are isomorphic to direct sums of uniform right ideals

<u>Proof:</u> (c) $\Rightarrow$ (b) $\Rightarrow$ (a)

(a) $\Rightarrow$ (c): $M = 0$ ✓ <u>$M \neq 0$</u>. Pick $P_R \leq M_R$ s.t. $M_R/P_R \cong U_R$, with $U_R \leq R_R$ uniform (3.1). U_R proj. $\Rightarrow M_R \cong P_R \oplus U_R$, and $\operatorname{udim} P = \operatorname{udim} M - 1$. The claim follows by induction on udim.

<u>Cor 3.4</u> If M_R, N_R are f.g. proj and $\operatorname{udim} M_R \leq \operatorname{udim} N_R$, then $\exists M'_R \leq N_R: \ M_R \cong M'_R$

<u>Proof:</u> M_R, N_R are direct sums of uniform right ideals. Uniform right ideals are subisomorphic (2.6) $\square$

Prop 3.5 The maps

$$\{\text{f.g. proj. right } R\text{-ideals}\} \longleftrightarrow \{\text{f.g. proj. left } R\text{-ideals}\}$$

$$I \xmapsto{\varphi} I^{-1} = (R :_\ell I) = I^*$$

$$J^{-1} = (R :_r J) \xleftarrow{\psi} J$$

are lattice anti-automorphisms (i.e. inclusion-reversing).

Induce bij. $\{\text{essential left } R\text{-ideals}\} \leftrightarrow \{X_R : R_R \subseteq X_R \subseteq Q_R,\ X_R \text{ f.g.}\}$

Moreover, if $R_R \leq X_R \leq Q_R$, X_R f.g. then

(1) $(X/R)_R$ has finite length

(2) $(X/R)_R$ semisimple $\Leftrightarrow {}_R(R/X^{-1})$ semisimple

(3) $(X/R)_R$ isotypic semisimple $\Leftrightarrow {}_R(R/X^{-1})$ isotypic semisimple

(M_R isotypic semisimple means: M_R is semisimple with all simple summands isomorphic)

~~$\exists d \in Q^\times: I_R \cong dI_R \subseteq R_R$, R hereditary & noetherian~~

Proof: I_R f.g. proj. $\Rightarrow I \cong I^{**}$ (canonically, 1.3(2)) $\Rightarrow I = (R :_r (R :_\ell I)) = (I^{-1})^{-1}$, so φ, ψ are bijective. If $I_R \subseteq I'_R$ and $x \in (I')^{-1} \Rightarrow xI \subseteq xI' \subseteq R$ $\Rightarrow x \in I^{-1}$, so φ, ψ are inclusion-reversing. As order anti-automorphisms of posets, they are lattice anti-automorphisms $(+, \cap \rightsquigarrow \cap, +)$

(1) $\mathcal{L} := \{\text{submodules of } X_R/R_R\} \underset{\text{lattice iso}}{\cong} \{\text{submodules of } {}_R R/{}_R X^{-1}\}^{op}$ (noetherian)

$\Rightarrow (X/R)_R$ noetherian & artinian $\Rightarrow (X/R)_R$ has finite length

(2) M_R semisimple $\Leftrightarrow \forall N_R \leq M_R\ \exists K \leq M_R: M_R = N_R \oplus K_R$

$\Leftrightarrow \forall N_R \leq M_R\ \exists K \leq M_R: M_R = N_R + K_R \wedge 0 = N_R \cap K_R$

And the last property is expressible in terms of the lattice structure on submodules & preserved by anti-iso.

(3) By (2) $M_R = (X/R)_R$ semisimple $\Leftrightarrow {}_R(R/X^{-1})$ semisimple.

M_R isotypic $\Leftrightarrow$ no nonzero proper submodule has a unique complement

[M_R not isotypic $\Rightarrow$ homogeneous components have unique complement

M_R isotypic: $M_R \cong S_R \oplus \dots \oplus S_R \cong S_R \oplus (s, s, \dots, s) \oplus X_1 \oplus \dots \oplus X_n$ ($X_i \cong S$)] $\square$

Lemma 3.6 (1) If $R_R \leq M_R \leq Q_R$ with M_R f.g., then M_R projective, $\lambda\left(\left(M/R\right)_R\right) < \infty$

(2) $\forall x \in Q$: $\lambda\left(\left(xR+R/R\right)_R\right) < \infty$

(3) $I_R \leq R_R$ essential $\Rightarrow \lambda\left(\left(R/I\right)_R\right) < \infty$ [„$\Leftarrow$" holds if $Q=R$, see 3.7(2) below]

(4) If $0 \neq A \trianglelefteq R \Rightarrow A_R \leq_e R_R$ and R/A is an ordinian ring!

Proof. (1), (3) by 3.5, for „$\Leftarrow$" in (3) use that R does not contain simple submodules

(1) $\Rightarrow$ (2) ✓

(4) If $A_R \cap I_R$ for $0 \neq I_R \leq R_R$, then $I_R A_R \subseteq I_R \cap A_R = 0$

↯ R prime $\Rightarrow A_R \leq_e R_R$.

So 3.5 $\Rightarrow R/A$ ordinian. □

Def. A ring S has restricted minimum condition (rmc-r) on the right if $(R/I)_R$ is ordinian for all $I_R \leq_e R_R$.

Lemma 3.7 (1) If M_R is f.g. torsion, then $\lambda(M_R) < \infty$

(2) Let $N_R \leq M_R$ be f.g. proj. Suppose $Q \neq R$. TFAE

(a) udim M = udim N

(b) $N_R \leq_e M_R$

(c) $\lambda(M/N) < \infty$

Proof. (1) Let $M_R = \langle m_1, \dots, m_s \rangle_R$, $m_i R \cong R/\mathrm{rann}(m_i)$

m_i torsion $\Rightarrow \mathrm{rann}(m_i) \cap R^{\circ} \neq \emptyset \overset{3.6(3)}{\Rightarrow} \lambda(m_i R) < \infty$

$\Rightarrow \lambda(M_R) < \infty$

(2) (a) $\Rightarrow$ (b) ✓ (b) $\Rightarrow$ (c) Show: M_R/N_R is torsion, then (1) implies (b) $\Rightarrow$ (c).

Let $m \in M$, $I := \mathrm{rann}(m+N) = \{r \in R : mr \in N\} < R_R$

Suppose $0 \neq J \leq R_R$. If $mJ = 0 \Rightarrow J \subseteq I \subseteq J \cap I \neq 0$

If $mJ \neq 0 \underset{N \leq_e M}{\Rightarrow} mJ \cap N \neq 0 \Rightarrow J \cap I \neq 0$.

So $I \leq_e R_R \Rightarrow I \cap R^{\circ} \neq \emptyset$

(c) $\Rightarrow$ (a) Suppose $\mathrm{udim}\, M < \mathrm{udim}\, N$

$\Rightarrow \exists\, 0 \neq X_R \leq M_R : \quad N \cap X = 0$,

By 3.3(c) we can take $X_R \cong U_R$ with $U_R \leq R_R$ uniform

But then U_R does not contain a minimal submodule (2.6(3)).

uses $Q \neq R$

$\Rightarrow \quad \infty = \lambda(U_R) < \lambda((M/N)_R)$ $\square$

Cor 3.8 (1) Every f.g. $M_R \in \mathrm{Mod}\text{-}R$ decomposes uniquely as a direct sum of a finite length module F_R and a projective module P_R.

(2) P_R decomposes as direct sum of uniform right ideals (not uniquely)

3.1 Overrings

R HNP ring, $Q = q(R)$, $Q \neq R$

Def: (1) An overring of R is a ring S s.t. $R \subseteq S \subseteq Q$

(2) An overring S is right finite if S_R is f.g.

($\Rightarrow S_R$ f.g. proj)

Then $q(S) = q(R) = Q$ and S is prime Goldie [Remark after Goldie's Thm]

Lemma 3.9 Let S be an overring of R

(1) S_R, ${}_RS$ are flat (i.e, $S_R \otimes -, - \otimes_R S$ are exact)

(2) $\mu : S \otimes_R S \to S$, $a \otimes b \mapsto ab$ is an iso

(3) $\forall M \in \mathrm{Mod}\text{-}S$: $M \otimes_R S \to M$, $m \otimes s \mapsto ms$ is an iso of S-modules

(4) — " — : M_R projective $\Rightarrow M_S$ projective

(5) $\forall I \leq S_S$: $I = (I \cap R)S \cong (I \cap R) \otimes_R S$

(6) Under restriction to S-modules: $\cong_R = \cong_S$, $\mathrm{Hom}_R = \mathrm{Hom}_S$, $\mathrm{Ext}^1_R = \mathrm{Ext}^1_S$, $\otimes_R = \otimes_S$.

Proof (1) If S_R is f.g., it is projective (3.5), and f.g. projective modules are flat. If S_R is arbitrary,

$$S_R = \bigcup \{X_R : R_R \leq X_R \leq S_R,\ X_R \text{ f.g.}\} = \varinjlim \{X_R : R_R \leq X_R \leq S_R\},$$

so S_R is a direct limit of flat modules. Since $\varinjlim$ commutes with $- \otimes_R M$, S_R is itself flat.

(2) Need to show $\ker(\mu)=0$. Suppose $x=\sum_{i=1}^{n} a_i \otimes b_i \in S\otimes_R S$ s.t. $\sum_{i=1}^{n} a_i b_i = 0$

$$\xrightarrow[\text{to } S\otimes -,\ -\otimes Q]{(1)} \quad S\otimes_R S \hookrightarrow S\otimes_R Q \hookrightarrow Q\otimes_R Q,$$

so we can consider $x\in Q\otimes_R Q$ and show $x=0$ there.

Let $d\in R^*$ s.t. $da_i\in R$ $\forall i\in\{1,..,n\}$.

$$\Longrightarrow x=\sum_{i=1}^{n} a_i\otimes b_i = d^{-1}\otimes \sum_{i=1}^{n}(da_i)b_i = d^{-1}\otimes d\underbrace{\sum_{i=1}^{n} a_i b_i}_{=0} = 0.$$

(3) - (6) are consequences of (1),(2): [LR, Prop 2.5, Prop 4.13]

(3) $M_S \cong M_S\otimes_S S_S \cong M_S\otimes_S (S\otimes_R S) \cong (M\otimes_S S)\otimes_R S \cong M\otimes_R S$.

(4) $M\oplus X_R = R_R^{(I)} \Longrightarrow \underbrace{(M\otimes_R S)}_{\cong M_S}\oplus (X\otimes_R S)\cong S_S^{(I)}$

(6) For Hom: Let $\varphi\in \mathrm{Hom}_R(M_S, N_S)$

$$\Longrightarrow \varphi\otimes id \in \mathrm{Hom}_S(\underset{\underset{M_S}{\cong}}{M\otimes_R S},\ \underset{\underset{N_S}{\cong}}{N\otimes_R S}) \Longrightarrow \varphi=\varphi\otimes id$$

$\cong$: ✓, $\otimes$: as in (3), Ext^1_R: [LR, Prop 4.13(iii)], same idea

(5) $(I\cap R)\otimes_R S\cong (I\cap R)S$ bec. μ is bijective.

$(I\cap R)S=I$: „$\subseteq$" ✓

„$\supseteq$": We show $I/_{I\cap R}\otimes_R S=0$, then $I=IS\subseteq (I\cap R)S$

Note $I/_{I\cap R}\cong_R {R+I}/_R \leq (S/_R)_R$

But: $0\to R\to S\to S/_R\to 0$ SES (in Mod-R), gives

$$\begin{array}{l} 0\to R\otimes_R S\to S\otimes_R S\to (S/_R)\otimes_R S\to 0 \quad \text{SES by (1)} \\ \quad\ \ \underset{S_S}{\cong} \xrightarrow{\ \ id\ \ } \underset{S_S}{\cong} \end{array}$$

so $(S/_R)\otimes_R S=0$! $\Longrightarrow (I/_{I\cap R})\otimes_R S=0$.

$\square$

A ring satisfying (1) & (2) is sometimes called a <u>localization</u> of R

Thm 3.10 Every overring S of R is HNP!

Proof: $R \subseteq S \subseteq Q \Rightarrow S$ prime Goldie & $q(S) = Q$

Let $I \leq S_S \Rightarrow I \cap R \leq R_R \Rightarrow I \cap R$ f.g. proj. (in Mod-R)

$$\overset{3.9}{\Longrightarrow} I = \underset{\substack{\uparrow \\ \text{f.g. as } S\text{-module}}}{(I \cap R)} S \cong \underset{\substack{\uparrow \\ \text{projective as } S\text{-module}}}{(I \cap R) \otimes_R S}$$

$\Rightarrow I$ f.g. proj. S-module $\Rightarrow S$ right hereditary, right noetherian

Symmetry: S is HNP. $\square$

Now we relate simple modules to overrings.

Lemma 3.11 Every simple right R-module is isomorphic to a submodule of $(Q/R)_R$ (uses $Q \neq R$!)

Proof: Let U_R be simple $\Rightarrow U_R \cong R/I$, $I \leq R_R$ maximal.

$\overset{3.6(3)}{\Longrightarrow} I \leq_e R_R$. Since I is f.g. proj, there exists X_R, n s.t.

$I \underset{R}{\oplus} X_R \cong R_R^n$. So $I \underset{R}{\oplus} X_R \leq_e \underbrace{R \underset{R}{\oplus} X_R}_{M_R}$, $\dfrac{R \oplus X}{I \oplus X} \cong U_R$.

$\exists d \in R^\cdot : dM_R \subseteq I \oplus X \overset{\text{identify}}{\cong} R_R^n \hookrightarrow Q_R^n$

$\Rightarrow M \cong d^{-1}(dM)$ embeds into Q_R^n s.t. $R_R^n \subseteq M_R$

$\Rightarrow U_R \hookrightarrow Q_R^n / R_R^n \overset{U \text{ simple}}{\Longrightarrow} U \hookrightarrow Q_R / R_R$.

Def: 1) Let Y be a set of isomorphism classes of simple right R-modules.

$$R(Y) := \{ x \in Q : \text{all composition factors of } \underset{\substack{\uparrow \\ \text{finite length by } 3.6(2)}}{\tfrac{xR+R}{R}} \text{ are in } Y \}$$

2) An overring S of R *kills* a simple M_R if $M_R \otimes S = 0$

Thm 3.12 (1) $R(Y)$ is a ring.

(2) $R(Y)$ kills precisely the simple modules in Y

(3) If S is an overring of R, then $S = R(Y)$ for a unique Y.

$Y = \{[M_R] : M_R$ composition factor of a f.g. submodule of $S/R\}$

Proof (1) Let $x, y \in R$

$\underline{x+y \in R}$: $$xR+R/R \oplus yR+R/R \twoheadrightarrow xR+yR+R/R \supseteq (x+y)R+R/R$$

So all composition factors of $(x+y)R+R/R$ are in Y

$\underline{xy \in R}$: $$yR+R/R \longrightarrow xyR+xR/xR \twoheadrightarrow xyR+xR+R/xR+R$$
$$\bar{y} \longmapsto \overline{xy}$$

$xyR+xR+R/xR+R$, $xR+R/R$ have (all) composition factors in Y

$\Longrightarrow$ $xyR+xR+R/R$ has comp. factors in Y.

$$\begin{array}{c} \cup\!\!\mid \\ xyR+R/R \end{array} \Rightarrow xy \in R(Y)$$

(2) M_R simple $\Rightarrow$ $M \cong yR+R/R$, $y \in Q$ (3.11)

„$\Leftarrow$" $[M] \in Y \Rightarrow y \in R(Y) =: S$, S localisation $\rightarrow$ $S/R \otimes_R S = 0 \Rightarrow M \otimes_R S = 0$.

„$\Rightarrow$": Suppose now M_R is simple & $M_R \otimes_R S = 0$

$\xRightarrow{-\otimes_R S \text{ exact}} yS+S/S = 0 \Rightarrow y \in S. \Rightarrow [M] \in Y$.

(3) Let $S \supseteq R$ be an overring, $Y := \{[M_R] : M_R \text{ simple}, M_R \otimes S = 0\}$

Claim: $S = R(Y)$, uniqueness then follows from (2)

$$x \in S \iff \left({}^{xR+R}/_R\right) \otimes_R S \cong {}^{xS+S}/_S = 0$$

$\iff$ all composition factors of ${}^{xR+R}/_R$ are killed by S

$\iff x \in R(Y)$ $\square$

Example: $R = \mathbb{Z}$, $Q = \mathbb{Q}$, simple modules: $\mathbb{Z}/p\mathbb{Z}$, p prime.

$x = \frac{a}{b}$, a, b coprime, $b > 0$, $b = p_1^{e_1} \cdots p_r^{e_r}$, p_i distinct, $e_i > 0$.

$\bar{x}\mathbb{Z} := x + \mathbb{Z}/\mathbb{Z}$

$$\overline{\frac{a}{p_1^{e_1} \cdots p_r^{e_r}}}\mathbb{Z} \geq \overline{\frac{a}{p_1^{e_1 - 1} p_2^{e_2} \cdots p_r^{e_r}}}\mathbb{Z} \geq \cdots \geq \overline{\frac{a}{p_2^{e_2} \cdots p_r^{e_r}}}\mathbb{Z} \geq \overline{\frac{a}{p_2^{e_2-1} p_3^{e_3} \cdots p_r^{e_r}}}\mathbb{Z} \geq \cdots \quad \bar{\mathbb{Z}}$$

gives a composition series, composition factors: $\mathbb{Z}/p_i\mathbb{Z}$, each w. multiplicity e_i.

So, for P a set of primes, $\langle P \rangle := \{q_1 \cdots q_r : q_i \in P\}$,

$Y = \{[\mathbb{Z}/p\mathbb{Z}] : p \in P\}$,

usual localization

$$\mathbb{Z}(Y) = \left\{\frac{a}{b} : a \in \mathbb{Z},\ b \in \langle P \rangle\right\} = (\langle P \rangle)^{-1}\mathbb{Z}$$

$P = \emptyset \iff \mathbb{Z}(Y) = \mathbb{Z}$, $P = \mathbb{P} \iff \mathbb{Z}(Y) = \mathbb{Q}$.

$P = \mathbb{P} \setminus \{p\} = \mathbb{Z}_{(p)} = \left\{\frac{a}{b} : p \nmid b\right\}$

P finite ... S-integers.

Thm 3.13 $S = R(Y)$

$U \otimes_R S = US$

(1) Let U_R be simple, $[U_R] \notin Y$. Then

 (i) $U \hookrightarrow U \otimes_R S$ as essential R-submodule, and $(U \otimes_R S)_S$ is simple, $\forall$

 (ii) If U_R' is simple, $U_R' \notin Y$, and $U_R \otimes_R S_S \cong_S U_R' \otimes S_S \Rightarrow U_R \cong U_R'$

(2) V_S simple $\Rightarrow \exists$ unique simple $[U_R]$ s.t. $V_S \cong U_R \otimes S$ and

$\mathrm{soc}_R(V_S) = U_R$, under the natural embedding $U_R \hookrightarrow_R (U_R \otimes S)_R$

socle

Proof: (1) Let $U_R \cong R/M$, $M_R \leq R_R$ maximal

(i) $U \hookrightarrow U \otimes_R S$: First show $MS \cap R = M$. Otherwise $MS \cap R = R$

$\Rightarrow R \subseteq MS \Rightarrow R/M \subseteq MS/M \overset{-\otimes_R S \text{ and}}{\Rightarrow} U \otimes_R S \subseteq (MS/M) \otimes_R S = 0$

So $MS \cap R = M$.

$\Rightarrow U \cong R/_M \cong (R+MS)/_{MS} \subseteq S/_{MS} = (R/_M) \otimes_R S \cong U \otimes_R S$

$U \subseteq U \otimes_R S$ is essential, i.e. $R/_M \leq S/_{MS}$ is essential. (as R-modules)

Let $a \in S \setminus MS. \Rightarrow aR+R/_R \cong R/_J$, $J = \{x \in R : ax \in R\}$

All composition factors of $R/_J$ are in $Y \Rightarrow JS = S$

Claim: $aJ \nsubseteq M$.

Suppose $aJ \subseteq M \Rightarrow aJS \subseteq MS \Rightarrow aS \subseteq MS \Rightarrow a \in MS$ ↯ □(Claim)

$\Rightarrow aR \cap R \nsubseteq M$, since $aJ \subseteq aR \cap R$

$M \subseteq (aR+M) \cap R \subseteq R \Rightarrow (aR+M) \cap R = R \Rightarrow aR+MS \supseteq R+MS \supsetneq MS$

$\Rightarrow aR+MS/_{MS} \supseteq R+MS/_{MS}$.

Simplicity:
So even $\underset{(1)}{R/_M} \cong R+MS/_{MS}$ is contained in every R-submodule of $S/_{MS}$!

$\Rightarrow S/_{MS} = S+MS/_{MS}$ contained in every S-submodule of $S/_{MS}$.

$\Rightarrow S/_{MS} \cong U \otimes_R S$ is simple

(1) $\Rightarrow U \subseteq US \subseteq U \otimes S \Rightarrow U \otimes S = US$

(ii) $U' \otimes S \cong U \otimes S \Rightarrow U, U'$ essential in $U \otimes S \overset{U,U' \text{ simple}}{\Rightarrow} U \cong U'$.

(2) $V_s \cong S/_N$, $N \leq S_s$ maximal. Let $a \in S \setminus N$.

$\Rightarrow aR+N/_N$ is finite length R-module [*]

$\Rightarrow \exists$ simple R-module $U \subseteq aR+N/_N \subseteq V_s$, $US = V_s$

$\Rightarrow [U] \notin Y$ [as $U \otimes_R S \twoheadrightarrow US \neq 0$]

$\Rightarrow U \otimes_R S$ simple by (1) $\Rightarrow U \otimes_R S \cong US = V$. □

[*: $R/_I \cong aR+N/_N$ with $I = \{r \in R : ar \in N\}$. Let $x \in N \cap Q^\times$.
Multiplying with an element of R° from the right, wlog $x \in R^\circ$.
$aR^\circ \cap xR \neq \emptyset$, so let $y \in R^\circ, z \in R$: $ay = xz \Rightarrow y \in I \cap R^\circ$
$\Rightarrow \lambda(R/_I) < \infty$ by 3.6(3))]

Prop 3.14 Let $S = R(Y)$, W_R simple with $[W_R] \notin Y$. Suppose $\mathrm{Ext}^1(X, W) = 0$ $\forall [X_R] \in Y$. Then

$$W_R \cong (W \otimes_R S)_R, \quad w \mapsto w \otimes 1$$

Proof: $W \leq_e W \otimes_R S$ (3.13). Suppose there exists $w \otimes x \in W \otimes_R S \setminus W$. Then $xR+R/R$ has all comp. factors in Y. $xR+R/R \longrightarrow (w\otimes x)R + W/W$, so $(w\otimes x)R+W/W$ has finite length & all comp. factors in Y. Let $M_R/W_R \subseteq (w\otimes x)R+W/W$ with $X_R := M_R/W_R$ simple. Then there is a SES

$$0 \to W_R \to M_R \to X_R \to 0$$

which is nonsplit, because $W_R \leq_e M_R$ ↯. □

3.2 Ideals & right finite overrings

Def: $I \lhd R$ is *invertible* if $\exists J \lhd R$: $R = IJ = JI$

Lemma Let $I \lhd R$ be invertible [Here it is enough for R to be prime Goldie]

(1) $(R :_\ell I) = \{x \in Q : IxI \subseteq I\} = (R :_r I) = I^{-1}$ ← $\boxed{\text{dangerous notation for two-sided ideals!}}$

[Let $x \in (R :_\ell I) \Leftrightarrow xI \subseteq R \Rightarrow IxI \subseteq I$; for „⇐" multiply by inverse from left]

(2) If $R = IJ = JI$, then $J = I^{-1}$

[$J \subseteq I^{-1}$ ✓, $R = IJ \subseteq II^{-1} \subseteq R \Rightarrow II^{-1} = R$; also $I^{-1}I = R$, uniqueness of inverses in monoids gives $I^{-1} = J$]

(3) $(I_R)^* = ({}_RI)^*$ using our identifications ← (2.5) [this is just (1) again]

(4) $O_\ell(I) = R = O_r(I)$ [$R \subseteq O_\ell(I)$ ✓ $\underline{O_\ell(I) \subseteq R}$:

Suppose $xI \subseteq I \Rightarrow x\underbrace{II^{-1}}_{=R} \subseteq \underbrace{II^{-1}}_{=R} \Rightarrow xR \subseteq R \Rightarrow x \in R$]

(5) If $J \leq R_R$,

$$\{\text{submodules of } (R/J)_R\} \xleftrightarrow{\text{bij}} \{\text{submodules of } (I/JI)_R\}$$

$$K/J \longleftrightarrow KI/JI \qquad [\text{multiplication by } I / I^{-1}]$$

(6) $\lambda(I^n/I^{n+1}) = \lambda(R/I) \quad \forall n \geq 1$ [by (5) with $J = A^n$]

(7) $tr(I_R) = (I_R)^* I_R = \sum_{f \in I^*} im(f) = R$ [by (2)]

<u>Aside on the meaning of (7):</u>

Let T be any ring.

<u>Prop 3.15</u> For $M \in Mod\text{-}T$ TFAE:

(a) $Hom_T(M, -) : \underline{Mod}\text{-}T \to \underline{Ab}$ is faithful, i.e.,

if $f : A_T \to B_T$, $f \neq 0 \Rightarrow Hom(M, f) : Hom(M, A) \to Hom(M, B)$, $\varphi \mapsto f \circ \varphi$ is nonzero

[Explicitly: $\exists g : M_T \to A_T$ s.t. $f \circ g \neq 0$]

(b) $tr(M) = T$

(c) T is a direct summand of some M^n

(d) — " — $M^{(I)}$

(e) every $A \in Mod\text{-}T$ is epimorphic image of some $M^{(I)}$

<u>Proof:</u> (a) $\Rightarrow$ (b) Suppose $tr(M) \subsetneq T$, let $\pi : T \to T/tr(M)$.

$\pi \neq 0 \Rightarrow \exists f \in M_T^* : \pi \circ f \neq 0 \Rightarrow f(M) \nsubseteq tr(M)$ ↯

(b) $\Rightarrow$ (c) $\exists f_1, \dots, f_n \in M^*, y_1, \dots, y_n \in M : 1 = \sum_{i=1}^n f_i(y_i)$

$$\varphi : \begin{cases} M^n \longrightarrow T \\ (x_1, \dots, x_n) \longmapsto \sum_{i=1}^n f_i(x_i) \end{cases} \quad \text{is split epi} \Rightarrow T \subseteq^{\oplus} M^n$$

$$(y_1 t, \dots, y_n t) \longleftarrow t$$

(c) $\Rightarrow$ (d) ✓ (d) $\Rightarrow$ (e) ✓

(e) $\Rightarrow$ (a) Let $0 \neq f : A_T \to B_T$, let $g : \bigoplus_{i \in I} M_i \twoheadrightarrow A$ ($M_i \cong M$)

$\Rightarrow f \circ g \neq 0 \Rightarrow \exists i \in I : f \circ g|_{M_i} \neq 0$ □

<u>Def:</u> M_T is a <u>generator</u> ($M^* M = R$) if it satisfies the conditions in (3.15).
It is a <u>progenerator</u> if it is also f.g. proj. ($MM^* = End(M_R)$).

<u>Morita Theory:</u> [L, Chapter 18]
Suppose M_T is a progenerator, $S := End(M_T)$. Then

$$\begin{array}{ccc} \underline{\text{Mod}}\text{-}T & \underset{\text{category equivalence}}{\sim} & \underline{\text{Mod}}\text{-}S \\ A_T & \longmapsto & \text{Hom}_T({}_S M_T, A_T)_S \\ B_S \otimes_S M_T & \longleftarrow & B_S \end{array}$$

Conversely, category equivalences between module categories are induced by progenerators!

<u>Exm</u> T ring $\Rightarrow$ T_T^n progenerator
induces equivalence $\text{Mod-}T \approx \text{Mod-}M_n(T)$

So T noetherian / ordinion / prime Goldie / hereditary / ...
$\Leftrightarrow M_n(T)$ — " —

—— (Aside ends)

So two-sided invertible ideals in HNP rings are progenerators!

<u>Prop 3.16</u> Let $M \triangleleft R$ be a nonzero ~~maximal~~ ideal
(1) If M is maximal, M is either invertible or idempotent ($M^2 = M$)
(2) If M is idempotent, then

$$(M_R)^* = O_\ell(M) \supsetneq R, \quad ({}_R M)^* = O_r(M) \supsetneq R,$$

$$O_\ell(M) \neq O_r(M)$$

(3) ideals maximal among idempotents are maximal ideals.

Proof: (1) $M \subseteq \underbrace{(M_R)^* M}_{= tr(M)} \subseteq R$, idempotent by (1.3) bec. M_R f.g. proj

Maximality $\Rightarrow$ $M = (M_R)^* M$ or $M^*_R M = R$

If $M = (M^*_R M) \Rightarrow M$ idempotent.

Otherwise $(M_R)^* M = R$ and symmetry implies $M ({}_R M)^* = R$ (bec. M not idempotent), so M invertible

(2) $Q \supseteq \underset{(R :_e M_R)}{\overset{\shortparallel}{(M_R)^*}} \supseteq \underset{(M :_e M_R)}{\overset{\shortparallel}{O_e(M)}}$ [only uses idempotence of M, max. not required]

let $x \in (M_R)^* \Rightarrow xM \subseteq R \Rightarrow$ $xM = xM^2 \subseteq RM \subseteq M \Rightarrow x \in (M_R)^*$,

so $(M_R)^* = O_e(M) = (R :_e M)$, analogous: $({}_R M)^* = O_r(M) = (R :_r M)$,

$M_R \subsetneq R \Longrightarrow R \subsetneq O_e(M)$ (3.5) , $O_r(M) \supsetneq R$

$(R :_e M) M \subseteq R$, but $\underset{({}_R M)^*}{\overset{\shortparallel}{(R :_r M)}} M = O_r(M)$ (dual basis lemma, 25)

$\Rightarrow (R :_e M) \neq (R :_r M)$, and $O_e(M) \neq O_r(M)$

(3) Suppose I is max. among idempotent ideals, but not maximal.

$\Rightarrow I \subsetneq M \subsetneq R$, M maximal ideal

$\Rightarrow M$ not idempotent $\Rightarrow M$ invertible $\Rightarrow \lambda(M^n / M^{n+1}) = \lambda(R/M)$

$\Rightarrow \lambda(R/M^n) = n\,\lambda(R/M) \geq n$

$I = I^n \Rightarrow \lambda(R/I) = \lambda(R/I^n) \geq \lambda(R/M^n) \geq n \ \forall n$ ↯.

(to $\lambda(R/I) < \infty$)

$\square$

Def. An overring $S \supseteq R$ is right finite if S_R is f.g. as right R-module.

Thm 3.17 There is an order-reversing bijection

$$\{\text{right finite overrings of } R\} \longleftrightarrow \{\overset{\text{non zero}}{\text{idempotent}} \text{ ideals of } R\}$$

$$S \longmapsto (S_R)^* = (R :_e S)$$

$$O_r(I) = (I :_r I) \longleftarrow I$$

Proof: Let $S \supseteq R$ be a right finite overring.

$\Rightarrow$ $(R :_e S) = \{q \in Q : qS \subseteq R\}$ is an ideal of R (bec. $RS \subseteq S$), and a right ideal of S ($S \cdot S \subseteq S$)

$\Rightarrow (S_R)^* = (R :_e S) = \underbrace{(R :_e S)}_{= S^*} S = tr(S) =: I$

I is idempotent (1.4) $\overset{3.16(2)}{\Longrightarrow}$ $O_r(I) = ({}_R I)^* = ((S_R)^*)^* = S_R$

Conversely, let $0 \neq I \lhd R$ be idempotent

$\Rightarrow$ $S = O_r(I)$ is an overring. Let $d \in I \cap R^\circ$

$\Rightarrow$ $S_R \cong dS_R \subseteq R_R \overset{R_R \text{ noeth.}}{\Longrightarrow} S_R$ f.g., so S is right finite overring

Finally, $\overset{3.16(2)}{\Longrightarrow}$ $S_R = O_r(I)_R = ({}_R I)^*$, so $S_R^* = ({}_R I)^{**} = I$ □

3.3 Asano $\overset{\text{(rings)}}{\text{orders}}$ and Dedekind prime rings

Let T be a prime Goldie ring.

Lemma 3.18 For $I \lhd T$ TFAE:

(a) I is invertible

(b) ${}_T I$, I_T are generators

(c) $O_r(I) = O_e(I) = T$ and ${}_T I$, I_T are f.g. projective

(d) — " — ${}_T I$, I_T are projective

Proof: (a) $\Leftrightarrow$ (b) ✓ $\left((I_T)^* I_T = T = \mathrm{tr}(I_T), \quad {}_TI\,({}_TI)^* = T = \mathrm{tr}({}_TI) \right)$

(a) $\Rightarrow$ (c) $\quad O_e(I) = T = O_r(I)$, see lemma $^{(*)}$ at beginning of §3.2

$\Rightarrow O_e(I) = T = I_T \underbrace{(T:_r I)}_{I^{-1}} \overset{(*)}{=} I_T \underbrace{(T:_e I)} = I_T (I_T)^*$

$\Rightarrow I_T$ f.g. proj. by dual basis lemma, similarly ${}_TI$ f.g. proj.

(c) $\Rightarrow$ (a): $\quad T = O_e(I) \underset{\text{dual basis lemma}}{=} I\,(T:_e I)$, similarly $(T:_r I)\,I = T$

(c) $\Rightarrow$ (d) ✓

(d) $\Rightarrow$ (c) By dual basis lemma, $\overset{(1.2)(1)}{\Longrightarrow} \exists\, a_i \in I_T,\ f_i \in (I_T)^*$, s.t.

$\forall x \in I$ only finitely many $f_i(x) \neq 0$, and $x = \sum_i' a_i f_i(x)$

Now let $d \in I \cap T^\circ$, $\quad d = \sum_{i=1}^{n}{}' a_i f_i(d)$ for some finite subfamily

$\mathrm{Hom}(I_T, T) = (T:_e I)$, so each f_i is left multiplication by some $b_i \in (T:_e I)$

$\Rightarrow \forall x \in I: \quad x = d(d^{-1}x) = \sum_{i=1}^{n}{}' a_i b_i \cancel{d} \cancel{d^{-1}} x = \sum_{i=1}^{n}{}' a_i f_i(x)$

$\xRightarrow[1.2(2)]{\text{dual basis lemm}} I_T$ is f.g. projective $\square$

Theorem 3.19 TFAE

(a) Every nonzero submodule of a (left or right) progenerator is a generator

(b) T is a maximal order (i.e., if $T' \supseteq T$ is overring, and $aT'b \subseteq T$ for some $a, b \in Q^\times$, then $T' = T$)

& each ideal is f.g. proj. as right & left module

(c) T is a max. order & each ideal is reflexive (i.e. $I_T^{**} \cong I_T$, ${}_TI \cong {}_TI^{**}$ canonically)

(d) Every $0 \neq I \trianglelefteq T$ is invertible

Def. T satisfying the conditions of 3.19 is an Asano order (or Asano ring)

<u>Proof</u>: (a) ⇒ (b) Suppose $T' \supseteq T$, $a, b \in Q^\times$: $aT'b \subseteq T$,

Say $a = a's^{-1}$, $b = b't^{-1}$, $a', b', s, t \in T^\bullet$

$\Rightarrow \quad a'T'b' \underset{T \subseteq T'}{\subseteq} a's^{-1}T'b' \subseteq Tt \subseteq T.$

Then $S := T + a'T' + Ta'T'$ is a ring with $\underline{T \subseteq S \subseteq T'}$

and $a'T' \subseteq S$, $Sb' \subseteq T$.

Let $I = (T :_r S) = \{ x \in q(T) : Sx \subseteq T \}$.

Then $0 \neq I \trianglelefteq T$ and $S \subseteq O_\ell(I)$. But I is a generator, so (3.18) $O_\ell(I) = T \Rightarrow S = T$.

Let $J = (S :_\ell T')$. Then $0 \neq J \trianglelefteq S = T$, and $T' \subseteq O_r(J) = T$, so $T = T'$!

<u>So</u>: T' is a max. order. By 3.18 every ideal is f.g. projective on both sides

(b) ⇒ (c) ✓ f.g. projective modules are reflexive (1.3)

(c) ⇒ (d) Let $0 \neq I \trianglelefteq T$. Let $J := (T :_\ell I) I \trianglelefteq T$ be the trace ideal of I_T.

$\forall x \in q(T)$: $q \in (T :_\ell J) \Rightarrow qJ \subseteq T \Rightarrow q(T :_\ell I) I \subseteq T \Rightarrow q(T :_\ell I) \subseteq (T :_\ell I)$

$\Rightarrow q \in O_\ell((T :_\ell I)) = T$ bec. T max. order $(O_\ell((T :_\ell I)) \sim T)$

$\Rightarrow (T :_\ell J) = T$, $\Rightarrow J \underset{J \text{ reflexive}}{=} (T :_r (T :_\ell J)) = T \Rightarrow I$ left invertible

By symmetry $I(T :_r I) = T$.

<u>(d) ⇒ (a)</u> Let A_T be a progenerator, $0 \neq B_T \leq A_T$.

$I := tr(B_T) = B_T^* B_T \trianglelefteq T$.

<u>Claim</u>: $I \neq 0$

[$B_T \subseteq T_T^n$ for some n, pick $0 \neq b \in B$, write $b = \sum_{i=1}^n b_i e_i$, wlog $b_1 \neq 0$

Then $p: T_T^n \to T$, proj. onto first component, gives $p(b) = b_1 \neq 0$

$\Rightarrow B^* B \neq 0$]

$\Rightarrow$ I invertible $\Rightarrow$ $(T :_e I)\, I = T \Rightarrow (T :_e I)\, B_T^{*} B_T = T$

$\Rightarrow$ $(T :_e I) B_T^{*} \subseteq B_T^{*}$, so $T \subseteq B_T^{*} B_T$. $\square$

<u>Cor. 3.20</u> TFAE for T

(a) Every nonzero submodule of a (left or right) progenerator is a progenerator

(b) T is an HNP ring and a max. order.

(c) T is an HNP ring and an Asano order.

<u>Proof</u>: (a) $\Rightarrow$ (b) T HNP, as every left/right ideal is f.g. proj.

T is a max order by 3.19(a)$\Rightarrow$(b). , (b) $\Rightarrow$ (a) ✓

(b) $\Leftrightarrow$ (c) Using (b) of 3.19 $\square$

<u>Def.</u> A ring satisfying the conditions in 3.20 is a <u>Dedekind prime ring</u> or <u>nc Dedekind domain</u> if it is a domain.

$$\begin{tikzcd} & & \{\text{hereditary prime rings}\} \arrow[d, dash] \\ \{\text{Asano orders}\} \arrow[dr, dash] & & \{\text{HNP rings}\} \arrow[dl, dash] \\ & \{\text{Dedekind prime rings}\} & \end{tikzcd}$$

<u>Cor 3.21</u> Let T be a commutative domain. TFAE

(a) every $0 \neq I \trianglelefteq T$ is invertible

(b) T is an Asano order

(c) T is an HNP ring

(d) T is hereditary

Such a ring is called a <u>Dedekind domain</u> ($\rightarrow$ algebraic number theory, affine nonsingular curves)

Proof: (a) $\Leftrightarrow$ (b), (b) $\Rightarrow$ (c) (every one-sided ideal is two-sided!), follow from T. 3.19, (c) $\Rightarrow$ (d) ✓,

(d) $\Rightarrow$ (c): Argue as in Lemma 3.18 (d) $\Rightarrow$ (c) (dual basis Lemma, applied twice)

(c) $\Rightarrow$ (a) Let $0 \neq I \trianglelefteq T \xRightarrow[\text{Lemma}]{\text{dual basis}} I(T:_e I) = O_e(I)$

By commutativity $(T:_e I) = (T:_r I)$, so $I(T_e : I) \subseteq T$

$\Rightarrow O_e(I) \subseteq T \Rightarrow I(T:_e I) = T$. $\square$

Some Examples:

(1) Simple Goldie rings are trivially Asano orders, $y_i x_i - x_i y_i = 1$

e.g. $A_n(K) = K[y_1, \dots, y_n][x_1, \frac{\partial}{\partial y_1}] - [x_n, \frac{\partial}{\partial y_n}]$, char $K = 0$

Can show gldim $A_n(K) = n$, so $A_n(K)$ for $n > 1$ is not hereditary (hered. $\Leftrightarrow$ gldim ≤ 1) [McR, 7.5.8]

(2) If R is prime Goldie, and each $I \triangleleft R$ is principal on both sides, then R is Asano.

Further: $I = aR = Rb \Rightarrow I = Ra = bR$

[Let $0 \neq I = aR = Rb \Rightarrow a, b \in R^\circ$

$(Ra^{-1})I = R = I(b^{-1}R) \Rightarrow I$ invertible & $Ra^{-1} = b^{-1}R$

$\Rightarrow bR = Ra \triangleleft R$ Ra, aR ideals $\Rightarrow Ra \subseteq aR$, $aR \subseteq Ra$]

E.g., $A_n(k)[z]$ [McR, 5.2.7, Exm (iii)]

(3) Let R be a prime PIR. Then R is a Dedekind prime ring.

[By (2) R is an Asano order. By ass. R is noetherian. Let $I \leq R_R$.

$\Rightarrow I \oplus J \leq_e R_R$ for some $J \leq R_R \Rightarrow I \oplus J \cong cR$ for some

$c \in R^\circ \Rightarrow cR_R \cong R_R$, so I is projective,

$\xRightarrow{\text{Symm.}}$ R HNP Asano order $\xRightarrow{3.20}$ R is Dedekind prime]

(4) Let K be a field of characteristic 0.

$B_1(K) = k(y)[x, \frac{d}{dy}]$ is Euclidean, hence a simple nc. Dedekind domain (by (3)).

$A_1(K)$ is HNP + Asano order (hereditary is harder to prove, cf [McR, 7.5.8]), hence also a simple Dedekind domain.

(5) R simple Artinian $\Longrightarrow$ $R[x]$ Dedekind prime ring

[$R = M_n(D)$, D div. ring $\Rightarrow$ $R[x] \cong M_n(D)[x] \cong M_n(D[x])$ + Morita invariance] (which we haven't proved)

(6) Let D be a commutative Dedekind domain, $K = q(D)$, $M \leq K^n_D$, then $\mathrm{End}(M_D)$ is a Dedekind prime ring [McR, 5.3.15]

(7) Let D, K be as in (6). Let Q be a central simple K-algebra ($\dim_K Q < \infty$, $Z(Q) = K$). A classical D-order in Q is a ring $D \subseteq \mathcal{O} \subseteq Q$ s.t. $\mathcal{O}_D$ is f.g., $K\mathcal{O} = Q$. If $\mathcal{O}$ is a maximal classical D-order (i.e., not properly contained in a proper larger such order), then $\mathcal{O}$ is a Dedekind prime ring. ([McR, 5.3.16], takes some work to prove)

$$\begin{tikzcd} & Q \cong M_n(D) \\ \mathcal{O} \arrow[ur, dash] & K \arrow[u, dash] \\ D \arrow[u, dash] \arrow[ur, dash] & \end{tikzcd}$$

e.g. $M_n(\mathbb{Z})$, $M_n(\mathbb{Z}[\sqrt{-5}])$,

Hurwitz quaternions: $\mathbb{Z}[i, j, k, \frac{\pm 1 \pm i \pm j \pm k}{2}]$, $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$

(8) While $S = A_1(K) = K[y][x; \frac{d}{dy}]$ is a Dedekind prime ring, $R := \mathbb{I}_S(xS) = K + xS$ is HNP but not a max. order, hence not an Asano order (xS is an idempotent R-ideal)

(9) $\begin{bmatrix} \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$, p prime is an HNP ring, but not a max. order.

Similarly, if D is a comm. Dedekind domain, $M \trianglelefteq D$ maximal,

$$\begin{bmatrix} D & M \\ D & D \end{bmatrix}$$ is a HNP ring, but not maximal order

<u>Thm 3.22</u> Let T be an Asano order. Every nonzero ideal factors <u>uniquely</u> as a <u>commutative</u> product of max. ideals. Fractional T-ideals form a free abelian group, generated by the maximal ideals.

<u>Proof</u>: Let $M_1 \neq M_2 \trianglelefteq T$ be maximal ideals. We show:

$$M_1 M_2 = M_1 \cap M_2 = M_1 M_2 \quad (*)$$

$$X := M_1^{-1}(M_1 \cap M_2) \Rightarrow X \trianglelefteq T \quad (\text{bec. } M_1 \cap M_2 \subseteq M_1)$$

$$M_1 X = M_1 \cap M_2 \subseteq M_2 \xrightarrow{M_2 \text{ prime}, M_1 \not\subseteq M_2} X \subseteq M_2$$

$$M_1 \cap M_2 = M_1 X \subseteq M_1 M_2 \subseteq M_1 \cap M_2 \Rightarrow M_1 X = M_1 M_2$$

$\xrightarrow{M_1 \text{ invertible}}$ $M_2 = X$, so $M_1 \cap M_2 = M_1 M_2$. $M_1 \cap M_2 = M_2 M_1$ by symmetry.

Thus the nonzero two-sided ideals form a monoid.

<u>Existence of factorizations</u>: Ideals are f.g. as right modules (3.19(b)), and so T satisfies ACC on two-sided ideals $\Rightarrow$ every nonempty set of two sided ideals contains maximal elements.

$\Omega := \{0 \neq I \trianglelefteq T : I$ is not product of max. ideals$\}$. Suppose $\Omega \neq \emptyset$
Let $I \in \Omega$ be maximal, let $M \supseteq I$ be a maximal ideal

$\Rightarrow M^{-1}I \trianglelefteq T$, $\quad M^{-1}I \supsetneq I$ by invertibility of I.

$\xRightarrow{I \in \Omega \text{ max}}$ $M^{-1}I = M_1 \cdots M_k$, $M_i \trianglelefteq T$ maximal ideals $\Rightarrow I = MM_1 \cdots M_k$

<u>Uniqueness</u>: $M \trianglelefteq T$ max $\Rightarrow$ M prime ideal, so $IJ \subseteq M \Rightarrow I \subseteq M$ or $J \subseteq M$

$\Rightarrow M$ is a prime element in the monoid of ideals, and the standard proof goes through

$\Big(I = M_1 \cdots M_n = N_1 \cdots N_e \Rightarrow M_j \subseteq N_1$ for some j, wlog $j = 1$,
$\xRightarrow{\text{max}}$ $M_1 = N_1 \Rightarrow M_2 \cdots M_n = N_2 \cdots N_e \;\; \ldots \rightarrow$ induction $\Big)$

<u>OR</u> The fancy way:

The fractional (two-sided) T-ideals form a group G that is lattice ordered wrt intersection $I \cap J$ and addition $I+J$.

$I \leq J \Rightarrow IK \leq JK, \; KI \leq KJ$, so G is an ℓ-group

If $\{I_j\}$ is a set of frac. T-ideals with lower bound I, then $\bigcap I_j \in G$,

if — " ——— upper bound J, then $\sum I_j \in G$,

and therefore abelian (see: Steinberg, Lattice ordered Rings and Modules, Thm 2.3.1 + 2.3.9) (actually in dual order)

It has ACC on integral elements, so every integral element is a product of atoms, the lattice structure shows there are gcds (and lcms) $\Rightarrow$ the monoid of integral elements is free abelian $\square$

Ideal Theory of Dedekind prime rings.

- If R is a Dedekind prime ring with $q(R)=Q$, $R \neq Q$, and $S \subseteq Q$ is a equivalent maximal order, then S is a Dedekind prime ring as well, and R, S are Morita equivalent [McR, 5.2.13]

Define a category $\mathcal{G}=\mathcal{G}(R)$:

- $\mathrm{Ob}\, \mathcal{G} = \{\text{maximal orders } S \text{ with } S \sim R\}$
- $\mathrm{Hom}_{\mathcal{G}}(R,S) = \{\text{fractional } (S,R)\text{-ideals } {}_S I_R\}$

$= \{\text{fractional right } R\text{- \& left } S\text{-ideals}\}$

${}_T J_S \circ {}_S I_R := {}_T J I_R = \{xy : x \in J, y \in I\}$

product in Q

- Identity in $\mathrm{Hom}_{\mathcal{G}}(R,R)$ is R.

Well-definedness easy to check.

Maximality of $R, S \Rightarrow O_\ell({}_S I_R) = S$, $O_r({}_S I_R) = R$

and $(R :_\ell I) = \{x \in Q : IxI \subseteq I\} = (S :_r I) \boxed{=: {}_R (I^{-1})_S}$

${}_S I_R \, {}_R (I^{-1})_S = {}_S I_R \, (R :_\ell I) = S$ (bec. I_R f.g. projective)

${}_R (I^{-1})_S \, {}_S I_R = (R :_\ell I) I = R$ (bec. I_R generator)

So every morphism in $\mathcal{G}$ is an isomorphism, i.e. $\mathcal{G}$ is a groupoid (the Brandt groupoid)

-) $\mathrm{Hom}_{\mathcal{G}}(R,R) = \mathrm{End}_{\mathcal{G}}(R) = \{\text{nonzero frac. } R\text{-ideals}\}$. By Thm 3.22, $\mathrm{End}_{\mathcal{G}}(R)$ is a free abelian group with basis the maximal ideals of R

-) $\mathcal{G}$ is connected, i.e., for every $S, T \in \mathrm{Ob}\,\mathcal{G}$, $\mathrm{Hom}_{\mathcal{G}}(S,T) \neq \emptyset$
 [one can check $TS \in \mathrm{Hom}(S,T)$]

-) Suppose $S, T \in \mathrm{Ob}\,\mathcal{G}$, ${}_TA_S \in \mathrm{Hom}_{\mathcal{G}}(S,T)$

$$\Rightarrow \varphi_A : \begin{cases} \mathrm{End}_{\mathcal{G}}(S) \xrightarrow{\sim} \mathrm{End}_{\mathcal{G}}(T) \\ {}_SI_S \longmapsto {}_TA_S I_S \, {}_S(A^{-1})_T \end{cases}$$

is a group iso with inverse $\varphi_{A^{-1}}$.

Note: This isomorphism does not actually depend on the choice of A!:

Let $A, B \in \mathrm{Hom}_{\mathcal{G}}(S,T)$

$${}_TA_S I_S A_T^{-1} = {}_TB_S I_S B_T^{-1} \Leftrightarrow {}_SB^{-1}_T A_S I_S A^{-1}_T B_S = I$$

$$\Leftrightarrow {}_S(B^{-1}A)_S \, I \, {}_S(B^{-1}A)_S^{-1} = I$$

$$\| $$

$(B^{-1}A)(B^{-1}A)^{-1} I$ by commutativity of $\mathrm{End}_{\mathcal{G}}(S)$.

So there are canonical isos $\Phi_{ST} : \mathrm{End}_{\mathcal{G}}(S) \to \mathrm{End}_{\mathcal{G}}(T)$ under which we may identify two-sided (fractional) ideals of S and of T.

-) For ${}_TI_S \in \mathrm{Hom}(S,T)$; $T = O_\ell(I)$, $S = O_r(I)$ by maximality,

so: $I \subseteq T \Leftrightarrow I \cdot I \subseteq I \Leftrightarrow I \subseteq S$

We say ${}_T I_S$ is integral $\Leftrightarrow$ $II \subseteq I$, let $\underline{\mathcal{G}_+ \subseteq \mathcal{G}}$ be the subcategory of all integral (one-sided) ideals

We'd hope that elements of $\mathcal{G}_+$ factor (in some sense) uniquely into maximal one-sided ideals!

Lemma 3.23: Let $I \in \mathrm{Hom}_{\mathcal{G}_+}(R,S)$.

(1) I is irreducible in $\mathcal{G}_+ \Leftrightarrow I = O_\ell(I) \Leftrightarrow I = O_r(I)$

(2) I is an atom in $\mathcal{G}_+$ (i.e., $I = JK$ in $\mathcal{G}_+$ implies J or K is a unit)

$\Leftrightarrow$ I max. right ideal in $O_r(I)$

$\Leftrightarrow$ I max. left ideal in $O_\ell(I)$

Proof: (1) We consider $I = O_r(I)$, „$\Leftarrow$" ✓

„$\Rightarrow$" If $I \subsetneq O_r(I) \Rightarrow \lambda(O_r(I)/I) > 0$

$\Rightarrow \forall J \in \mathrm{Hom}_{\mathcal{G}_+}(\cdot, O_\ell(I))$: $JI \subseteq I \Rightarrow \lambda\left(O_r(I)/_{JI}\right) > 0$

$\Rightarrow JI \neq O_r(I)$.

(2) Consider right ideals, $R = O_r(I)$

„$\Leftarrow$": Suppose $I = KJ \Rightarrow I_R \leq J_R \leq R_R$. By maximality $J = R$ or $I = J$.

Case $J = R$: ✓

Case $I = J$: $I = KI \Rightarrow S = II^{-1} = KII^{-1} = K \Rightarrow K$ unit

„$\Rightarrow$" Suppose $I \subsetneq J \subsetneq R$.

$\Rightarrow I = (IJ^{-1})J$ and $IJ^{-1} \subsetneq O_\ell(J)$ as $J \neq I$.

Lemma 3.24 Every $I \in \mathrm{Hom}_{\mathcal{G}_+}(S,R)$ has a factorization

$I = M_1 \cdots M_k \qquad k \geq 0$

with atoms M_i of $\mathcal{G}_+$ s.t. $O_\ell(M_1) = O_\ell(I)$, $O_r(M_k) = O_r(I)$, $O_r(M_i) = O_\ell(M_{i+1})$.

In any such representation, $k = \lambda\left((R/I)_R\right) = \lambda\left({}_S(S/I)\right)$.

Proof: $\lambda((R/I)_R)$, $\lambda((S/I)_S)$ are finite (3.6(3)), so we can use induction on the length: If $I=R$, there is nothing to show. Otherwise, let $I \leq_R M \leq_R R_R$ be a max. right ideal $\Rightarrow \lambda(R/M^{-1}I) = \lambda(R/I) - 1$, so $IM^{-1} = M_2 \cdots M_k$ with atoms M_i and $k = \lambda(R/I) - 1$. $\square$

If $I_R = M_1 \cdots M_k = N_1 \cdots N_k$ with atoms M_i, N_i then

$$R_R \geq M_k \geq M_{k-1}M_k \geq \dots \geq I_R \quad \text{and} \quad R_R \geq N_k \geq N_{k-1}N_k \geq \dots \geq I_R$$

yield composition series of $(R/I)_R$, and the Jordan-Hölder Theorem implies

$$M_i \cdots M_k / M_{i-1}M_i \cdots M_k \cong N_{\sigma(i)} \cdots N_k / N_{\sigma(i)-1}N_{\sigma(i)} \cdots N_k \quad \text{as } R\text{-modules}$$

for some permutation σ.

We'd like something a bit nicer, that only involves the i-th and $\sigma(i)$-th factor.

<u>Def</u> (1) $I \in \mathrm{Hom}_{G^+}(R,S)$ is <u>transposable</u> to $J \in \mathrm{Hom}_{G^+}(R',S')$ if $\exists C \in \mathrm{Hom}_{G^+}(R,R')$ s.t.

$${}_{S'}J_{R'}C_R = I_R \cap C_R \quad \text{and} \quad I_R + C_R = R_R \qquad \text{(right } R\text{-modules)}$$

(2) I, J are <u>projective to each other</u> (lattice-theory terminology) if there is a sequence $I = K_0, \dots, K_n = J$ of integral one-sided ideals s.t. (I_i, I_{i+1}) or (I_{i+1}, I_i) are transposable for all i.

<u>I.e.</u> perspectivity is an equivalence relation, the symmetric & transitive closure of transposability.

<u>Remark</u>: If I, J, C are as in (1), then

$$\underline{J}C = I \cap \underline{C} = \underline{D}I \quad \text{for } D \in \mathrm{Hom}_{G^+}(S,S')$$

and C is transposable to D

Thm 3.25: If $I \in \mathrm{Hom}_{G_+}(R,S)$, and

$I = M_1 \cdots M_k = N_1 \cdots N_k$ in G_+ with atoms M_i, N_i, then there exists a permutation σ s.t. M_i and $N_{\sigma(i)}$ are projective to each other for $1 \leq i \leq k$. (let us say: the factorizations are equivalent)

Proof: By induction on k.

If $M_k = N_k$, then $M_1 \cdots M_{k-1} = N_1 \cdots N_{k-1}$ and the claim follows by IH. So suppose $M_k \neq N_k$. Since M_k, N_k are max. right ideals, $M_k + N_k = R$.

$$\begin{tikzcd}[column sep=small]
 & R \arrow[dl, dash] \arrow[dr, dash] & \\
M_k \arrow[dr, dash] & & N_k \arrow[dl, dash] \\
 & M_k \cap N_k \arrow[d, dotted, dash] & \\
 & I &
\end{tikzcd}$$

Now $M_k / M_k \cap N_k \cong R/N_k$ (as right R-modules) is simple. $\Rightarrow$ $M_k \cap N_k = \underline{N_k'} M_k = M_k' N_k$ for atoms M_k', N_k'.

Then N_k is transposable to N_k', and M_k is transposable to M_k'.

Now $I (M_k \cap N_k)^{-1} = L_1 \cdots L_{k-2}$ with atoms L_i

$\Rightarrow$ $I = L_1 \cdots L_{k-2} N_k' M_k = L_1 \cdots L_{k-2} M_k' N_k$.

By (IH) $M_1 \cdots M_{k-1}$ and $L_1 \cdots L_{k-2} N_k'$ are equivalent, and

so $M_1 \cdots M_k \sim L_1 \cdots L_{k-2} N_k' M_k \sim L_1 \cdots L_{k-2} M_k' N_k$

$\underset{\text{also by (IH)}}{\sim} N_1 \cdots N_{k-1} N_k$.

$\square$

Cor 3.26: To go from any factorization of I to any other one, it suffices to successively replace two neighboring factors by two new ones.

Proof: (Sketch) By the induction in the previous proof:

We replace $N_k' M_k$ by $M_k' N_k$, which is of the claimed type. The rest follows by induction.

$\square$

Def: A prime Goldie ring R is <u>bounded</u> if for every $I \leq_e R_R$ there exists $0 \neq J \trianglelefteq R$ s.t. $J \subseteq I$, and for every $I \leq_{e\,R} R$, there exists $0 \neq J \trianglelefteq R$ s.t. $J \subseteq I$

Exm: $M_n(\mathbb{Z})$ is bounded (ideals are of form $M_n(d\mathbb{Z})$), while $A_1(K)$ is not if char $K=0$.

Classical orders are always bounded.

If a Dedekind prime ring R is bounded, then so is every $S \sim R$.

Recall: $\mathrm{End}_{\mathcal{G}}(R) \cong \mathrm{End}_{\mathcal{G}}(S)$ canonically, i.e., two-sided ideals of R and S can be identified. Given $I \trianglelefteq R$ let (I) be the equivalence class of all frac. ideals identified with I, and $(I)_S$ the unique frac. S-ideal in the class.

$\mathbb{G} := \{ (I) : I \text{ frac. } R\text{-ideal} \}$ is a free abelian group,

$\mathbb{G}_+ := \{ (I) : 0 \neq I \trianglelefteq R \}$ free abelian submonoid on classes of max. ideals.

$\boxed{\text{Now: Let } R \text{ be a bounded Dedekind prime ring}}$

Def: For $I \in \mathrm{Hom}_{\mathcal{G}_+}(R,S)$, let $A \trianglelefteq R$ be maximal s.t. $A \subseteq I$ and define the <u>lower bound</u> $\Phi(I) := (A) \in \mathbb{G}_+$.

$\Leftrightarrow A = \mathrm{ann}((R/I)_R)$

Lemma 3.27: (1) $\Phi(I) = (B)$ with $B \trianglelefteq S$ maximal s.t. $B \subseteq I$

(2) I atom $\Longrightarrow$ $\Phi(I)_R$ maximal ideal (converse false!)

(3) If $J \in \mathrm{Hom}_{\mathcal{G}_+}(S,T)$: $\Phi(J)\,\Phi(I) \leq \Phi(JI) \leq \Phi(J) \wedge \Phi(I)$

lcm in free abelian monoid.

induced by containment

(4) Let $I \in \mathrm{Hom}_{\mathcal{G}_+}(R,S)$, $J \in \mathrm{Hom}_{\mathcal{G}_+}(R',S')$ be projective to each other.

If I is an atom, then so is J and $\Phi(I) = \Phi(J)$

<u>Proof:</u> (1) We have $\mathrm{End}_{\mathcal{G}}(R) \xrightarrow{\sim} \mathrm{End}_{\mathcal{G}}(S)$, $A \mapsto IAI^{-1}$.
Since $R \mapsto II^{-1} = S$, this induces $\mathrm{End}_{\mathcal{G}^+}(R) \xrightarrow{\sim} \mathrm{End}_{\mathcal{G}^+}(S)$.

Let $0 \neq A \trianglelefteq R$. $A \leq I \iff IA \leq II \iff IAI^{-1} \leq III^{-1} = I$
& the claim follows

(2) It suffices to prove that $\Phi(I)_R$ does not factor as product of two proper ideals. Suppose $\Phi(I)_R = AB$,
$0 \neq A, B \underset{\neq}{\trianglelefteq} R$

$\Rightarrow I_R + A_R = R$ by maximality of I once $A \not\leq I$.

$$I \supseteq AB + IB = (A+I)B = RB = B \quad ↯ \text{ to } \Phi(I) = AB.$$

<u>Converse false:</u> in $R = M_2(\mathbb{Z})$, $\Phi\left(\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} R\right) = M_2(p\mathbb{Z}) = \Phi\left(\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} R\right)$
$\uparrow$ not atom.

(3) $J \supseteq \Phi(J)_T \Rightarrow {}_S C_T = J^{-1}\Phi(J)_T$ is integral.

$$\underline{\Phi(J)_T \, \Phi(I)_T} = JC\,\Phi(I)_T = JC\left(C^{-1}\Phi(I)_S C\right)$$

$$= {}_T J_S \, \Phi(I)_S \, C_T \underset{\substack{\uparrow \\ C \text{ integral,} \\ \text{both } T\text{-ideals}}}{\subseteq} {}_T J_S \, \Phi(I)_S \leq \underline{{}_T J_{SS} I_R}.$$

$\Rightarrow$ $\underline{\Phi(J)\,\Phi(I) \leq \Phi(JI)}$

$\Phi(JI) \leq \Phi(J) \wedge \Phi(I)$ follows bec. $\Phi(JI) \leq \Phi(J)$
& $\Phi(JI) \leq \Phi(I)$

(4) <u>First:</u> If I is transposable to J then I atom $\iff$ J atom

<u>Here:</u> $I_R + C_R = R_R$, ${}_{R'}J C_R = I \cap C$

$$\begin{tikzcd}[column sep=small] & R \arrow[dl, dash] \arrow[dr, dash] & \\ I \arrow[dr, dash] & & C \arrow[dl, dash] \\ & JC & \end{tikzcd}$$

$\rightarrow$ $(R/I)_R \cong (C/JC)_R$

$\Rightarrow$ $\lambda((R/I)_R) = \lambda((C/JC)_R) \underset{\substack{\uparrow \\ C \text{ invertible in } \mathcal{G}}}{=} \lambda((R'/J)_{R'})$

so I atom $\iff \lambda((R/I)_R) = 1 \iff \lambda((R'/J)_{R'}) = 1 \iff J$ atom

Now suffices to show: If I, J are atoms and I is transposable to J, then $\Phi(I) = \Phi(J)$.

Again $I_R + C_R = R_R$, $\quad J_R C_R = I_R \cap_R C_R$

$P := \Phi(I)_R \Rightarrow {}_{R'}C_R P C^{-1} = \Phi(I)_{R'}$

${}_{R'}C_R P_R \subseteq P_R \cap C_R \subseteq I \cap C = JC \Rightarrow CPC^{-1} \subseteq J_{R'}$

$\Rightarrow \Phi(J)_{R'} \supseteq CPC^{-1} \Rightarrow \Phi(J)_{R'} \geq \Phi(I)_{R'}$ But

$\Phi(I)_{R'}$ is a max. ideal & $\Phi(J)_{R'}$ is proper $\Rightarrow \Phi(J)_{R'} = \Phi(I)_{R'}$

$\Rightarrow \Phi(J) = \Phi(I)$. $\square$

Cor 3.28 Suppose R is a bounded Dedekind prime ring, $\mathcal{G} = \mathcal{G}(R)$. If $I = M_1 \cdots M_k = N_1 \cdots N_k$ in $\mathcal{G}_+$ with atoms M_i, N_i, then there exists a permutation σ s.t. $\Phi(M_i) = \Phi(N_{\sigma(i)})$ for all $1 \leq i \leq k$.

Proof: By Thm 3.25, there exists a permutation σ s.t. M_i and $N_{\sigma(i)}$ are projective to each other. By Lemma 3.27(4), $\Phi(M_i) = \Phi(N_{\sigma(i)})$. $\square$

Lemma 3.29 („Meta commutation", „Transposition")
Let $M \in \mathrm{Hom}_{\mathcal{G}_+}(R, S)$, $N \in \mathrm{Hom}_{\mathcal{G}_+}(S, T)$ be atoms. If $\Phi(M) \neq \Phi(N)$, then there exist unique $S' \in \mathrm{Ob}\,\mathcal{G}$, $N' \in \mathrm{Hom}_{\mathcal{G}_+}(R, S')$, $M' \in \mathrm{Hom}_{\mathcal{G}_+}(S', T)$ s.t.

$$N \cdot M = M' \cdot N' \quad \text{and} \quad \Phi(M) = \Phi(M'), \quad \Phi(N) = \Phi(N')$$

Proof: Define $M' := {}_T N M_R + \Phi(M)_T$ (left T-ideal).

Claim: M' is an atom and $M' \cap N = NM$.

[Proof of Claim: Note $\Phi(M) \nsubseteq \Phi(N)$ as $\Phi(M) \neq \Phi(N)$, and both correspond to maximal ideals. Therefore $\Phi(M) \nsubseteq N$.

We make use of the modularity law for left T-modules:

A, B, C left T-modules, $\overset{B \leq A}{\cancel{A \leq B}} \Rightarrow A \cap (B+C) = B + (A \cap C)$

$${}_T\Phi(M)_T N \leq {}_T\Phi(M)_T \cap {}_T N \overset{\text{bec. } \Phi(M)_T \nleq N}{\lneq} \Phi(M)_T \quad \text{(left } T\text{-modules)}$$

Maximality of N gives $\Phi(M) N = \Phi(M) \cap N$. Now:

$$M' \cap N = (NM + \Phi(M)_T) \cap N \overset{\text{modularity}}{=} NM + (\Phi(M)_T \cap N) = NM + \Phi(M)_T N$$

$$= NM + (NN^{-1})\Phi(M)_T N = NM + {}_T N_S (N^{-1}\Phi(M)_T N)$$

$$= {}_T NM_R + {}_T N_S \Phi(M)_S = N(M + \Phi(M)) = NM.$$

Thus $NM = M'X$ for some integral X. Since NM is a factorization of length 2, it suffices to show ${}_T NM \subsetneq {}_T M' \subsetneq T$.

By contradiction.

Case 1: $NM = M' \Rightarrow \Phi(M)_T \subseteq {}_T M' = {}_T NM \subseteq {}_T N$ ↯.

Case 2: $T = M' \Rightarrow NM = T \cap N = N \Rightarrow M = R$ ↯.

□ (Claim)

Existence: Because ${}_T NM \leq {}_T M' \lneq T$, there exists $N' \in \mathrm{Hom}_{G_+}(R, O_r(M'))$ $NM = M'N'$. Because length of factorizations in G_+ are unique & M' is an atom, also N' must be an atom.

Now $\Phi(M)_T \subseteq M' \subsetneq T$ by construction. Because $\Phi(M)_T$ is a maximal ideal of T (Lemma 3.27(2)), necessarily $\Phi(M)_T = \Phi(M')_T$. Then Cor 3.28 shows $\Phi(N') = \Phi(N)$.

Uniqueness: Suppose ${}_T NM_R = {}_T M''_T N''_R$ with atoms $M'' N''$ s.t. $\Phi(M'') = \Phi(M)$, $\Phi(N'') = \Phi(N)$.

$\Rightarrow \underbrace{{}_T NM + \Phi(M)_T}_{={}_T M'} \subseteq {}_T M'' \subsetneq T$. By maximality of M', we get $M'' = M'$, and then $N' = N''$ and $T = T'$

□

Cor 3.29 Suppose $I = M_1 \cdots M_k$. Let σ be any permutation of $\{1, \dots, k\}$. Then there exists a factorization $I = N_1 \cdots N_k$ s.t. $\Phi(N_i) = \Phi(M_{\sigma(i)})$ for $1 \leq i \leq k$.

Proof. Write σ as a product of transpositions and apply Lemma 3.29 repeatedly. $\square$

Remark: In general $\Phi(IJ) \neq \Phi(I)\Phi(J)$. But by 3.28 we can define a function $\eta : G_+ \to \mathbb{G}_+$ ("abstract norm") by

$\eta(I) = \Phi(M_1) \cdots \Phi(M_k)$ if $I = M_1 \cdots M_k$ with atoms M_i

η can be extended to a groupoid hom. $G \to \mathbb{G}$.

Example 1) $R = M_n(\mathbb{Z})$. R is a PIR, $a \in R^\bullet \Leftrightarrow \det(a) \neq 0$, $a \in R^\times \Leftrightarrow \det(a) \in \{\pm 1\}$, ideals: $M_n(d\mathbb{Z})$, $d \in \mathbb{Z}$.

Every right R-ideal is of form aR with $a \in R^\bullet$.

Then $O_\ell(aR) = aRa^{-1} \cong R$ (as rings)

$\Rightarrow \operatorname{Ob} \mathcal{G} = \{ aRa^{-1} : a \in R^\bullet \}$

note: $aRa^{-1} = R \Leftrightarrow aR = Ra$
$\Leftrightarrow a$ gen. R-ideal $\Leftrightarrow a = ud$, $u \in GL_n(\mathbb{Z})$, $d \in \mathbb{Q}^\times$

$$\operatorname{Hom}_{\mathcal{G}}(R, bRb^{-1}) = \{ aR : aRa^{-1} = bRb^{-1} \}$$
$$= \{ aR : a = bud,\ u \in GL_n(\mathbb{Z}),\ d \in \mathbb{Q}^\times \}$$

$\underbrace{ab}R = a(bRb^{-1}) \underset{\uparrow \text{ product in } \mathcal{G}}{\cdot} bR$

$\underline{\Phi(aR) = ?}$ $\quad a = (a_{ij})$, $e := \operatorname{lcm}(a_{ij}) \Rightarrow M_n(e\mathbb{Z}) = \Phi(aR)_R$

$a \in bRb^{-1}$, $a(bRb^{-1})$ atom $\underset{\uparrow \text{ Smith normal form}}{\Leftrightarrow}$ $\det(a)$ prime $\Leftrightarrow$ $\Phi(a(bRb^{-1}))_{bRb^{-1}}$ prime ideal.

$\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} R$, $\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} R$, ..., $\begin{bmatrix} p & p-1 \\ 0 & 1 \end{bmatrix} R$ $\leftarrow$ $p+1$ atoms of norm p.

pR

$\eta(a(bRb^{-1})) = (\det(a) R)$

2) $R = \mathcal{H}$ similar: PID, so all equivalent max. orders are conjugate

For p odd prime: $\hat{R}_{(p)} \cong M_2(\mathbb{Z}_p) \to p+1$ atoms of norm pR

$\underline{p=2}$: $\hat{R}_{(2)}$ local ring, unique atom $(1+i)R$

unique fact in $G_+ \triangleq$ unique fact of elements

3) $K = \mathbb{Q}(\sqrt{3})$, $\left(\frac{-1,-1}{K}\right)$, $\mathcal{O} = \mathbb{Z}[\sqrt{3}]$

$R = {}_{\mathbb{Z}[\sqrt{3}]}\langle 1, i, \sqrt{3}i+j, \sqrt{3}+k\rangle$ is max. order ($\to$ MAGMA)

$\Rightarrow$ There exist two isomorphism classes of right R-ideals,

$[R]$, $[I]$, I non-principal.

$\mathcal{O}_\ell(I) \cong {}_{\mathbb{Z}[\sqrt{3}]}\langle 1, i, 2-\sqrt{3}i+j, \sqrt{3}+2i+k\rangle$

4. Modules over Dedekind prime rings

Let R be a Dedekind prime ring with quotient ring $Q = \varrho(R)$, $Q \neq R$

<u>Lemma 4.1</u> (Schanuel's Lemma) Let S be a ring, and suppose there are SES

$$0 \to K_S \to P_S \xrightarrow{\pi} M_S \to 0 \qquad 0 \to K'_S \to P'_S \xrightarrow{\pi'} M_S \to 0$$

with P_S, P'_S projective. Then $K \oplus P' \cong K' \oplus P$

<u>Proof</u>: Let $L = \{(p,p') \in P \oplus P' : \pi(p) = \pi'(p')\}$.

There is a SES

$$0 \to L \hookrightarrow P \oplus P' \longrightarrow M \to 0$$
$$(p,p') \longmapsto \pi(p) - \pi'(p')$$

$L \twoheadrightarrow P$, $(p,p') \mapsto p$ is surjective w. kernel

$\{(0,p') : \pi'(p') = 0\} \cong \ker(\pi') = K'$. Since P is projective,

$P \oplus K' \cong L$. Symmetrically, $L \cong P' \oplus K$. $\square$

Prop 4.2 Let I, J, K be f.g. proj. R-modules with $\operatorname{udim} I = \operatorname{udim} J$ and $K \neq 0$

$\Rightarrow \exists X_R : \quad I \oplus K \cong J \oplus X.$

Proof: $\operatorname{udim} J = \operatorname{udim} I \overset{L.3.7.}{\Longrightarrow}$ wlog $I \leq_e J \overset{3.7}{\Longrightarrow} \lambda(J/I) < \infty$,

say $n = \lambda(J/I)$.

Induction on n.

$n=0$: $I \cong J$, so $X = K$ works.

$n \geq 1$, $n-1 \to n$: Let $I \leq I' \leq J$ be such that I'/I is simple.

Because R is a Dedekind prime ring, K is a (pro)generator.

$\overset{3.19}{\Longrightarrow} \exists$ epi $K^{(\Omega)} \to I'/I \Rightarrow \exists$ nonzero hom $\varphi : K \to I'/I$.

Because I'/I simple, φ is epi. Have SES:

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \ker(\varphi) \arrow[r] & K \arrow[dr, "\varphi"] & & \\
& & & I'/I \arrow[r] & 0 \\
0 \arrow[r] & I \arrow[r] & I' \arrow[ur] & &
\end{tikzcd}$$

$\overset{\text{Schanuel}}{\Longrightarrow} \quad I' \oplus \ker(\varphi) \cong K \oplus I$

Now, $\lambda(J/I') = n-1$, so by IH $\exists X'$: $I' \oplus \underset{\substack{\neq 0 \\ \text{bec. } R \neq Q}}{\ker(\varphi)} \cong J \oplus X$.

So $I \oplus K \cong I' \oplus \ker(\varphi) \cong J \oplus X$.

$\square$

Cor 4.3 Let M_R be a f.g. proj. R-module.

Then $M_R \cong R^n \oplus I$ for some $n \geq 0$ and $I_R \leq R_R$.

Proof: We have $M_R \cong U_1 \oplus \dots \oplus U_s$ with uniform $U_i \leq R_R$

by Thm 3.3(c). Grouping $\operatorname{udim} R_R$ of these together at a time,

$M_R \cong I_1 \oplus \dots \oplus I_n \oplus J$ with $I_j \leq_e R_R$, $J \leq R_R$.

Since udim I_j = udim R, (4.2) gives
$I_1 \oplus I_2 \cong R \oplus I_2'$ for some $I_2' \leq R_R$.
The claim follows by induction.

Example: This is in general false for HNP rings that are not Dedekind prime rings!
[If $I \trianglelefteq_{\neq} R$ is idempotent, consider $M = I \oplus_R I_R$.
Then $MI = M$. But $(R \oplus J)I = I \oplus JI \neq R \oplus J$]
(*)

Cor 4.4 Let $I \leq R_R$.
(1) If $I \leq_e R_R$, there exists $J \leq_e R_R$ s.t. $I \oplus J \cong R_R^2$.
In particular, I is 2-generated.
(2) If I is not essential in R_R, there exist $J \leq R_R$ s.t.
$I \oplus J \cong R_R$, and I is cyclic.

Proof: (1) ✓ by (4.2)
(2) Let $n := \mathrm{udim}(I) < \mathrm{udim}\, R_R$.
Since R_R is a direct sum of uniform right ideals (Thm 3.3(c)),
we can write $R_R = J_R \oplus K_R$ with $\mathrm{udim}(J) = k$, $K \neq 0$
$\overset{(4.2)}{\Longrightarrow} \exists X : \; I \oplus X \cong J \oplus K \cong R_R$. □

(*) Remark: The ideal I is in general not "unique" up to isomorphism.
E.g., one can show that for $R = A_1(K)$, char $K = 0$,
$I \oplus R \cong R \oplus R$ for all right ideals $0 \neq I \leq R_R$,
but there are non-principal right ideals (see [McR, 7.11.6 and 7.11.8])

Thm 4.5 If $M \oplus X \cong M \oplus Y$ with f.g. proj. X_R, Y_R, M_R and $\operatorname{udim} X = \operatorname{udim} Y \geq 2$, then $X \cong Y$.

So: $R^{n-1} \oplus I \cong R^{n-1} \oplus J \Rightarrow R \oplus I \cong R \oplus J$

(W/o proof, This is a consequence of a more general cancellation theorem of Bass, [McR, 11.7.13, 11.7.14])

Def: Projective M_R, N_R are <u>stably isomorphic</u> if
$\exists n \geq 0: R^n_R \oplus M_R \cong R^n_R \oplus N_R \iff R_R \oplus M_R \cong R_R \oplus N_R$
$\iff X \oplus M_R \cong X \oplus N_R$ (for some f.g. proj. $X \neq 0$)
Write $\langle M \rangle$ for stable isomorphism classes.

Prop 4.6: $G(R) := \{ \langle I \rangle \mid I$ right R-ideal$\}$ is an abelian group, the <u>(Steinitz) class group</u> with $0 = \langle R \rangle$ and $\langle I \rangle + \langle J \rangle := \langle K \rangle \iff I \oplus J \cong R \oplus K$

Proof: (4.2) guarantees existence of $\langle K \rangle$ & (4.5) its uniqueness
Associativity can be checked using (4.5);
Inverses: $\langle I \rangle + \langle J \rangle = \langle R \rangle \iff I \oplus J \cong R_R \oplus R_R$, so (4.2) gives existence of inverses $\square$

<u>„Trick"</u>: Let $(G, 0, +)$ be an abelian group, fix $z \in G$.
With $g \tilde{+} h := g + h - z$ also $(G, z, \tilde{+})$ is an abelian group, and $(G, 0, +) \xrightarrow{\sim} (G, z, \tilde{+})$, $g \mapsto g + z$.
So we can change the neutral element up to iso.
Inverse of g wrt. to $\tilde{+}$ is $\boxed{-g + 2z}$

Thm 4.6 Let V be the monoid of stable isomorphism classes of f.g. proj. R-modules, with addition induced from the direct sum. Then

$$V \cong \{ (n,g) \in \mathbb{N}_0 \times G(R) \ : \ n>0 \ \underline{\text{or}} \ g=0 \}$$

Proof: Fix $U \leq R_R$ uniform, let $d := \operatorname{udim} R_R$.
Use the trick to change the group structure on $G(R)$ to $\tilde{+}$ to assume wlog $\underset{\substack{\shortparallel \\ cl(U^d)}}{\langle U^d \rangle} = 0$ in $G(R)$.

Let $\langle M \rangle \in V$. Let $k \geq 0$ s.t. $\operatorname{udim}(M) + k \geq d+1$

$\overset{(4.2)}{\Longrightarrow} \exists l \geq 0 \ \exists I \leq_e R_R : \quad M \oplus U^k \cong U^l \oplus U \oplus \underline{I}$

Map $\langle M \rangle \longmapsto (\operatorname{udim}(M), cl(I))$

[Suppose $\langle M \rangle = \langle N \rangle \Rightarrow U \oplus M \cong U \oplus N$, $\quad \operatorname{udim} M = \operatorname{udim} N$

say $N \oplus U^{k'} \cong U^{l'} \oplus U \oplus I'$, $\quad$ wlog $k' \geq k$

$\Rightarrow \quad U^{l'+2} \oplus I' \cong N \oplus U^{k'+1} \cong M \oplus U^{k'+1} \cong U^{l+1+(k'-k)} \oplus I$

$\overset{(4.5)}{\Longrightarrow} U \oplus I' \cong U \oplus I \Rightarrow cl(I') = cl(I)$]

Additivity: say $\quad M \oplus U^k \cong U^{l+1} \oplus I, \quad N \oplus U^{k'} \cong U^{l'+1} \oplus I$

$\Rightarrow \quad (M \oplus N) \oplus U^{k+k'} \cong U^{l+l'+2} \oplus I \oplus J$

$\cong U^{l+l'+2+d} \oplus U^d + K \qquad (4.2)$

$\overset{4.5}{\Longrightarrow} \quad I \oplus J \cong U^d \oplus K \quad \Rightarrow \quad \langle K \rangle = \langle I \rangle \tilde{+} \langle J \rangle$ with modified group law.

Injective: By (4.5)

Surjective: Let $n > 0$, I right R-ideal,

$\overset{4.2}{\Longrightarrow} \exists M, k, l : \ M \oplus U^k \cong I \oplus U^l \quad \Rightarrow \quad \langle M \rangle$ maps to $\langle I \rangle$.

$\square$

Immediate consequence: $K_0(R) \cong \mathbb{Z} \times G(R)$

Thm 4.7 Let $0 \neq A \trianglelefteq R$. Then

(1) $\forall\, I \leq_e R_R$: $I/IA \cong R/A$

(2) R/A is an ordinary principal ideal ring.

Proof: (1) Since A invertible, $\lambda(A/IA) = \lambda(R/I) < \infty$ by beginning of sec. 3.2 ((5)). $\xrightarrow{\text{Dig}} \lambda(I/IA) = \lambda(R/A)$

$$\begin{tikzcd}[column sep=small, row sep=small]
R \arrow[d, dash] & & \\
I+A \arrow[d, dash] \arrow[dr, dash] & & \\
I \arrow[dr, dash] & A \arrow[d, dash] & \\
& A\cap I \arrow[d, dash] & \\
& IA &
\end{tikzcd}$$

Case 1: $A = M$ is a maximal ideal

R/A simple ordinary, I/IA R/A-module with

$\lambda(R/A) = \lambda(I/IA) \implies R/A \cong_{R/A} I/IA \implies R/A \cong_R I/IA$.

Case 2: $A = M^n$, $n \geq 1$, M maximal ideal.

$J(R/M^n) = M/M^n$

↑ Jacobson radical.

We have $\dfrac{IM}{M^n} = \left(\dfrac{I}{IM^n}\right) \cdot \left(\dfrac{M}{M^n}\right)$

$$R/M \underset{\substack{\uparrow \\ \text{Case 1}}}{\cong} I/IM \cong \frac{(I/IM^n)}{(IM)/(IM^n)} \cong \frac{\overbrace{(I/IM^n)}^{=:B}}{(I/IM^n)\underbrace{(M/M^n)}_{\bar M}}, \quad \bar R := R/M^n$$

So this module (B) is cyclic, i.e.

$B = \bar c \bar R + B \bar M$ for some $c \in R$.

Since $\bar M = J(\bar R)$, $\boxed{\text{Nakayama's lemma}}$ applies to show $B = \bar c \bar R$

$\implies I/IM^n$ is cyclic. $\implies R/M^n \twoheadrightarrow I/IM^n$

$\implies R/M^n \xrightarrow{\sim} I/IM^n$ (because the modules have the same length)

Case 3 $A = M_1^{e_1} \cdots M_n^{e_n}$, M_i maximal ideals, $e_i \geq 1$.

$\Rightarrow R/A \cong R/M_1^{e_1} \oplus \dots \oplus R/M_n^{e_n}$ (bec. $M_i + \prod_{j \neq i} M_j = R$, CRT applies)

$\Rightarrow I/IA \cong I/IM_1^{e_1} \oplus \dots \oplus I/IM_n^{e_n}$

By Case (2) each $I/IM_i^{e_i}$ is cyclic over $R/M_i^{e_i}$

$\rightarrow I/IA$ cyclic over R/A

$\Rightarrow$ (as in (2)) $R/A \cong I/IA$.

(2) We already know that R/A is artinian as left R-module & as right R-module (because it has finite length; cf. L. 3.6(4))

Let $I \leq R_R$ with $A \subseteq I$ $\Rightarrow I \leq_e R_R$

$\overset{(1)}{\Rightarrow} I/IA$ cyclic over R/A

But $I/IA \twoheadrightarrow I/A$, so I/A is also cyclic. $\square$

Thm 4.8 (Steinitz) Let R be a commutative Dedekind domain, $0 \neq M_R$ torsion-free. Let $Q = Q(R)$ be the quotient field of R.

(1) $M \cong I_1 \oplus \dots \oplus I_n$ for $0 \neq I_i \trianglelefteq R$, $n = \operatorname{udim} M_R$

(2) $M \cong R^{n-1} \oplus I$ with $I = I_1 \cdots I_n$

(3) If $M \cong R^{n-1} \oplus I \cong R^m \oplus J$ with $0 \neq I, J \trianglelefteq R$, $m \geq 0$, then $m = n-1$ and $J \cong I$ ($\Leftrightarrow J = cI$, $c \in Q^*$)

Proof: (1) By Thm 3.3(c)

(2) By induction it suffices to show $I \oplus J \cong R \oplus IJ$.

By Thm 4.7(1), $I/_{IJ} \cong R/_J$. Consider the SES

$$\begin{array}{ccccccccc} 0 & \to & IJ & \to & I & \to & I/_{IJ} & \to & 0 \\ & & & & & & \wr\wr & & \\ 0 & \to & J & \to & R & \to & R/_J & \to & 0 \end{array}$$

$\xRightarrow{\text{Schanuel's lemma}} IJ \oplus R \cong I \oplus J$

(3) Recall: $\mathrm{Hom}(\bigoplus_{i \in I} A_i, \prod_{j \in J} B_j) \cong \prod_{i,j} \mathrm{Hom}(A_i, B_j)$ for arbitrary modules

$m = n-1$ ✓ by udim

So: Can think of $\mathrm{Hom}_R(\underbrace{R^{n-1} \oplus I}_{M_R}, \underbrace{R^{n-1} \oplus J}_{N_R})$ as formal matrices

$$\begin{bmatrix} \mathrm{Hom}(R,R) & \cdots & \mathrm{Hom}(R,R) & \mathrm{Hom}(I,R) \\ \vdots & & \vdots & \vdots \\ \mathrm{Hom}(R,R) & \cdots & \mathrm{Hom}(R,R) & \mathrm{Hom}(I,R) \\ \mathrm{Hom}(R,J) & \cdots & \mathrm{Hom}(R,J) & \mathrm{Hom}(J,I) \end{bmatrix}$$

$\wr\wr$ (Lemma 2.5) $\leftarrow$ $\mathrm{Hom}(J,I) = (I:J)$

$$\begin{bmatrix} R & \cdots & R & I^{-1} \\ \vdots & & \vdots & \vdots \\ R & \cdots & R & I^{-1} \\ J^{-1} & \cdots & J^{-1} & (I:J) \end{bmatrix} \subseteq M_n(Q)$$

$(I:J) = IJ^{-1}$

R Dedekind (!)

$(I:J) = IJ^{-1}$ ["$\supseteq$" ✓ "$\subseteq$" $(I:J)J \subseteq I \Rightarrow (I:J)JI^{-1} \subseteq II^{-1} = R$
$\Rightarrow (I:J) \subseteq IJ^{-1}$]

So an iso is given by $A \in M_n(K)$ of such form, with inverse A^{-1} in a similar module with J & I swapped.

Note (Leibniz-formula): $\det(A) \in IJ^{-1}$, $\det(A^{-1}) \in JI^{-1}$
$\det(A)\det(A^{-1}) = 1$

Now: $\underline{J} = \det(A)\det(A^{-1})J \subseteq \underline{\det(A^{-1})I} \subseteq \underline{J}$

$\Rightarrow J = \underbrace{\det(A^{-1})}_{\in Q^\times} I$, so $J \cong I$ $\square$

Remark: Every f.g. M_R is uniquely a product of a torsion-free module and its torsion submodule $\text{tor}(M_R)$ (by Prop 3.2). Thm 4.8 describes the torsion free part, the torsion modules are direct sums of indecomposable cyclic modules (of the form R/p^e for $e \geq 1$, p a maximal ideal).